2 Elementary viscous flow

2.1. Introduction

Steady flow past a fixed aerofoil may seem at first to be wholly accounted for by inviscid flow theory. The streamline pattern seems right, and so does the velocity field. In particular, the fluid in contact with the aerofoil appears to slip along the boundary in just the manner predicted by inviscid theory. Yet close inspection reveals that there is in fact no such slip. Instead there is a very thin *boundary layer*, across which the flow velocity undergoes a smooth but rapid adjustment to precisely zero—corresponding to *no slip*—on the aerofoil itself (Fig. 2.1). In this boundary layer inviscid theory fails, and viscous effects are important, even though they are negligible in the main part of the flow.

To see why this should be so we must first make precise what we mean by the term 'viscous'. To this end, consider the case of simple shear flow, so that $\mathbf{u} = [u(y), 0, 0]$. The fluid immediately above some level y = constant exerts a stress, i.e. a force per unit area of contact, on the fluid immediately below, and vice versa. For an inviscid fluid this stress has no tangential component τ, but for a viscous fluid τ is typically non-zero. In this book we shall be concerned with *Newtonian* viscous fluids, and in this case the shear stress τ is proportional to the velocity gradient $\mathrm{d}u/\mathrm{d}y$, i.e.

$$\tau = \mu \frac{\mathrm{d}u}{\mathrm{d}y}, \tag{2.1}$$

where μ is a property of the fluid, called the *coefficient of viscosity*. Many real fluids, such as water or air, behave according to eqn (2.1) over a wide range of conditions (although there are many others, including paints and polymers, which are non-Newtonian, and do not; see Tanner (1988)).

From a fluid dynamical point of view the so-called *kinematic viscosity*

$$\nu = \mu/\rho \tag{2.2}$$

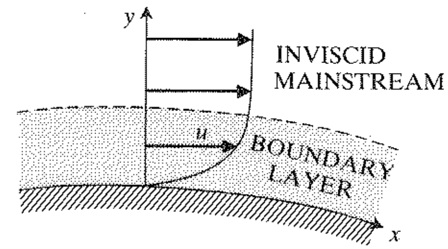


Fig. 2.1. A boundary layer.

is often more significant than μ itself, and some typical values of ν are given in Table 2.1. These values can vary quite substantially with temperature, but throughout much of this book we shall concentrate on a simple model of fluid flow in which μ, ρ, and ν are all constant.

We can see now, in general terms, why viscous effects become important in a boundary layer. The reason is that the velocity gradients in a boundary layer are much larger than they are in the main part of the flow, because a substantial change in velocity is taking place across a very thin layer. In this way the viscous stress (2.1) becomes significant in a boundary layer, even though μ is small enough for viscous effects to be negligible elsewhere in the flow.

But why are boundary layers so important that we begin this chapter with them? The answer is that in certain circumstances

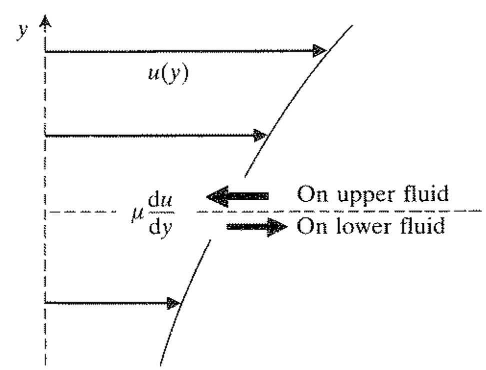


Fig. 2.2. Viscous stresses in a simple shear flow.

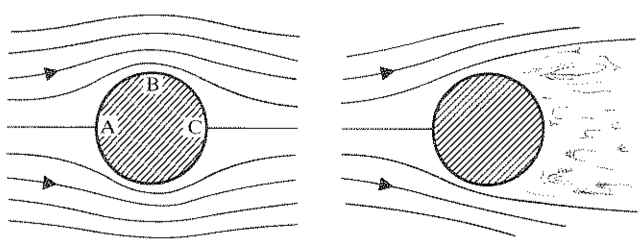
Table 2.1. Kinematic viscosity ν ($cm^2\,s^{-1}$) at 15°C.

Water	0.01	($\mu = 0.01$ c.g.s. units)
Air	0.15	($\mu = 0.0002$ c.g.s. units)
Olive oil	1.0	
Glycerine	18	
Golden syrup/treacle	~1200	($\nu \sim 200$ at 27°C)

they may *separate* from the boundary, *thus causing the whole flow of a low-viscosity fluid to be quite different to that predicted by inviscid theory.*

Consider, for example, the flow of a low-viscosity fluid past a circular cylinder. In the first instance it is natural to assume that viscous effects will be negligible in the main part of the flow, which will therefore be irrotational, by the argument of §1.5. If we solve the problem of irrotational flow past a circular cylinder (§4.5) we obtain the streamline pattern of Fig. 2.3(a). This 'solution' is not wholly satisfactory, for it predicts slip on the surface of the cylinder. We might then suppose that a thin viscous boundary layer intervenes to adjust the velocity smoothly to zero on the cylinder itself. But this turns out to be wishful thinking; the observed flow of a low-viscosity fluid past a circular cylinder is, instead, of an altogether different kind, with massive separation of the boundary layer giving rise to a large vorticity-filled *wake* (Fig. 2.3(*b*)).

Why does separation occur? The answer lies in the variation of pressure p along the boundary, as predicted by inviscid theory.



(*a*) (*b*)

Fig. 2.3. Flow past a circular cylinder for (*a*) an inviscid fluid and (*b*) a fluid of small viscosity.

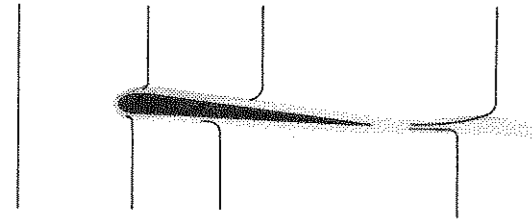


Fig. 2.4. Flow past an aerofoil: the fate of successive lines of fluid particles.

In Fig. 2.3(*a*), inviscid theory predicts that p has a local maximum at the forward stagnation point A, falls to a minimum at B, then increases to a local maximum at C, with $p_A = p_C$. This implies that between B and C there is a substantial increase in pressure along the boundary in the direction of flow. It is this *severe adverse pressure gradient along the boundary* which causes the boundary layer to separate, for reasons which are outlined in §§8.1 and 8.6 (see especially Fig. 8.2.)

An aerofoil, on the other hand, is deliberately designed to avoid such large-scale separation, the key feature being its slowly tapering rear. In Fig. 1.9, for example, the substantial fall in pressure over the first 10% or so of the upper surface is followed by a very *gradual* pressure rise over the remainder. For this reason the boundary layer does not separate until close to the trailing edge, and there is only a very narrow wake (Fig. 2.4). This state of affairs persists as long as the angle of attack α is not too large; if α is greater than a few degrees, the pressure rise over the remainder of the upper surface is no longer gradual,

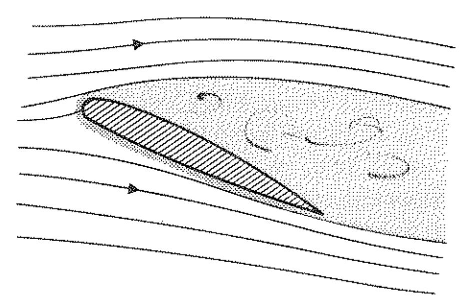


Fig. 2.5. Separated flow past an aerofoil.

large-scale separation takes place, and the aerofoil is said to be *stalled*, as in Fig. 2.5. This is the explanation for the sudden drop in lift in Fig. 1.11.

The most important overall message of this introduction is that *the behaviour of a fluid of small viscosity μ may, on account of boundary layer separation, be completely different to that of a (hypothetical) fluid of no viscosity at all.* From a mathematical point of view, what happens in the limit $\mu \to 0$ may be quite different to what happens when $\mu = 0$.

2.2. The equations of viscous flow

So far we have considered the motion of fluids of small viscosity. Yet there is more to the subject than this, including the opposite extreme of very viscous flow (Chapter 7). It is time, then, to take a more balanced—if brief—look at viscous flow as a whole.

The Navier–Stokes equations

Suppose that we have an incompressible Newtonian fluid of constant density ρ and constant viscosity μ. Its motion is governed by the *Navier–Stokes equations*†

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{g}, \qquad \nabla \cdot \boldsymbol{u} = 0. \tag{2.3}$$

These differ from the Euler equations (1.12) by virtue of the viscous term $\nu\nabla^2\boldsymbol{u}$, where ∇^2 denotes the Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

The no-slip condition

Observations of real (i.e. viscous) fluid flow reveal that both normal and tangential components of fluid velocity at a rigid boundary must be equal to those of the boundary itself. Thus if the boundary is at rest, $\boldsymbol{u} = 0$ there. The condition on the tangential component of velocity is known as the *no-slip condition*, and it holds for a fluid of any viscosity $\nu \neq 0$, no matter how small ν may be.

† The Navier–Stokes equations are derived from first principles in Chapter 6.

The Reynolds number

Consider a viscous fluid in motion, and let U denote a typical flow speed. Furthermore, let L denote a characteristic length scale of the flow. This is all somewhat subjective, but in dealing with the spin-down of a stirred cup of tea, for instance, 4 cm and 5 cm s^{-1} would be reasonable choices for L and U, while 10 m and 100 m s^{-1} would not. Having thus chosen a value for L and for U we may form the quantity

$$R = \frac{UL}{\nu}, \tag{2.4}$$

which is a pure number known as a *Reynolds number.*

To see why R should be important, note that derivatives of the velocity components, such as $\partial u/\partial x$, will typically be of order U/L—assuming, that is, that the components of $\boldsymbol{u}$ change by amounts of order U over distances of order L. Typically, these derivatives will themselves change by amounts of order U/L over distances of order L, so second derivatives such as $\partial^2 u/\partial x^2$ will be of order U/L^2. In this way we obtain the following order of magnitude estimates for two of the terms in eqn (2.3):

$$\begin{aligned} &\text{inertia term:} && |(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}| = O(U^2/L), \\ &\text{viscous term:} && |\nu\nabla^2\boldsymbol{u}| = O(\nu U/L^2). \end{aligned} \tag{2.5}$$

Provided that these are correct we deduce that

$$\frac{|\text{inertia term}|}{|\text{viscous term}|} = O\left(\frac{U^2/L}{\nu U/L^2}\right) = O(R). \tag{2.6}$$

The Reynolds number is important, then, because it can give a rough indication of the relative magnitudes of two key terms in the equations of motion (2.3). It is not surprising, therefore, that high Reynolds number flows and low Reynolds number flows have quite different general characteristics.

High Reynolds number flow

The case $R \gg 1$ corresponds to what we have hitherto called the motion of a fluid of small viscosity. Equation (2.6) suggests that viscous effects should on the whole be negligible, and flow past a thin aerofoil at small angle of attack provides just one example where this is indeed the case. Even then, however, viscous effects become important in thin boundary layers, where the unusually large velocity gradients make the viscous term much larger than the estimate in eqn (2.5). We show in §§8.1 and 8.2 that the typical thickness δ of such a boundary layer is given by

$$\delta/L = O(R^{-\frac{1}{2}}). \tag{2.7}$$

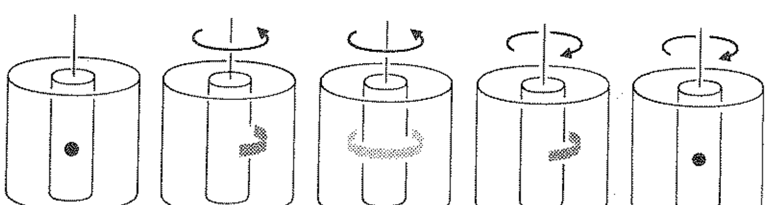
The larger the Reynolds number, then, the thinner the boundary layer.

A large Reynolds number is *necessary* for inviscid theory to apply over most of the flow field, but it is not sufficient. As we have seen, boundary layer separation can lead to a quite different state of affairs. A further complication at high Reynolds number is that steady flows are often *unstable* to small disturbances, and may, as a result, become *turbulent*. It was in fact in this context that Reynolds first employed the dimensionless parameter that now bears his name (see §9.1).

Low Reynolds number flow

Consider a laboratory experiment in which golden syrup occupies the gap between two circular cylinders, the inner one rotating and the outer one at rest. For reasonable rotation rates of the inner cylinder the Reynolds number might be in the region of 10^{-2} or so; it will certainly be much less than 1. At such Reynolds numbers there is no sign of turbulence, and the flow is extremely well ordered.

The flow is so well ordered, in fact, that if the rotation of the inner cylinder is stopped after a few revolutions, and the inner cylinder is then rotated back through the correct number of turns to its original position, a dyed blob of syrup, which has been greatly sheared in the meantime, will return almost exactly to its original configuration as a concentrated blob (Fig. 2.6).



(*a*) (*b*) (*c*) (*d*) (*e*)

Fig. 2.6. The reversibility of a very viscous flow.

This near *reversibility* is characteristic of low Reynolds number flows, and helps account, in fact, for the unusual swimming techniques that are adopted by certain biological micro-organisms such as the Spermatozoa (§7.5).

2.3. Some simple viscous flows: the diffusion of vorticity

We now turn to some elementary exact solutions of the Navier–Stokes equations. There is, in addition, a major theme running through §§2.3 and 2.4, and that theme is the *viscous diffusion of vorticity*, an important mechanism which was wholly absent in Chapter 1, where ν was zero.

Plane parallel shear flow

Suppose that a viscous fluid is moving so that relative to some set of rectangular Cartesian coordinates

$$\boldsymbol{u} = [u(y, t), 0, 0]. \tag{2.8}$$

Such a flow is termed a plane parallel shear flow. It automatically satisfies $\nabla \cdot \boldsymbol{u} = 0$, as u is independent of x, and in the absence of gravity† the Navier–Stokes equations (2.3) become, in component form:

$$\begin{gathered} \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \\ \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0. \end{gathered} \tag{2.9}$$

The pressure p is thus a function of x and t only. But from eqn (2.9) $\partial p/\partial x$ is equal to the difference between two terms which are independent of x. Thus $\partial p/\partial x$ must be a function of t alone. As we shall see shortly, there are important circumstances in which this fact enables us to deduce that $\partial p/\partial x$ must be zero.

† See footnote on p. 9.

First, however, it is instructive to see how eqn (2.9) may be obtained by a simple and direct application of the expression (2.1).

An ad hoc derivation of the equations of motion for a viscous fluid in plane parallel shear flow

First note that in the absence of viscous forces the corresponding Euler equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{2.10}$$

may be deduced by considering an element of fluid of unit length in the z-direction and of small, rectangular cross-section in the x–y plane, with sides of length δx and δy (see Fig. 2.7). The net pressure force on the element in the x-direction is

$$p(x)\,\delta y - p(x+\delta x)\,\delta y \doteqdot -\frac{\partial p}{\partial x}\,\delta x\,\delta y,$$

and this is equal to the product of the element's mass $\rho\,\delta x\,\delta y$ and its acceleration

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x},$$

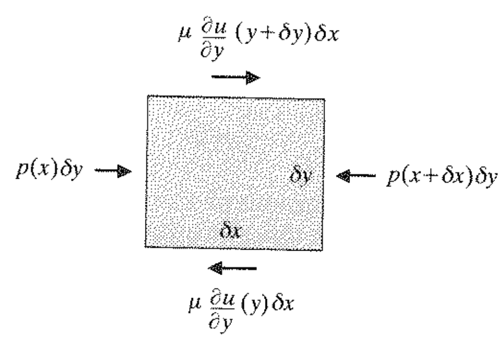


Fig. 2.7. The forces in the x-direction on a small rectangular blob in a plane parallel shear flow.

which reduces simply to $\partial u/\partial t$ because u is independent of x.

In a similar manner we may use eqn (2.1) to deduce that viscous forces on the top and bottom of the element give rise to a net contribution in the x-direction of

$$\mu \left.\frac{\partial u}{\partial y}\right|_{y+\delta y} \delta x - \mu \left.\frac{\partial u}{\partial y}\right|_{y} \delta x \doteqdot \mu \frac{\partial^2 u}{\partial y^2}\,\delta x\,\delta y, \tag{2.11}$$

whence eqn (2.10) becomes modified to

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2},$$

i.e. to eqn (2.9).

This equation is, of course, valid only for a very restricted class of flows, but the brevity of the above derivation does have its merits. In particular, it brings out rather clearly, via eqns (2.1) and (2.11), why the viscous term in the equation of motion (2.3) involves the *second* derivatives of the velocity field.

The flow due to an impulsively moved plane boundary

Suppose that viscous fluid lies at rest in the region $0 < y < \infty$ and suppose that at $t = 0$ the rigid boundary $y = 0$ is suddenly jerked into motion in the x-direction with constant speed U. By virtue of the no-slip condition the fluid elements in contact with the boundary will immediately move with velocity U. We wish to find how the rest of the fluid responds.

It is natural to look for a flow of the form (2.8), and eqn (2.9) then applies. We assume that the flow is being driven only by the motion of the boundary, i.e. not by any externally applied pressure gradient. This experimental consideration corresponds to asserting that the pressures at $x = \pm\infty$ are equal, and as $\partial p/\partial x$ is independent of x (so that p is a linear function of x) it follows that $\partial p/\partial x$ is zero.

The velocity $u(y, t)$ thus satisfies the classical one-dimensional *diffusion equation*

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{2.12}$$

together with the initial condition

$$u(y, 0) = 0, \qquad y > 0,$$

and the boundary conditions

$$u(0, t) = U, \quad t>0, \qquad u(\infty, t) = 0, \quad t>0.$$

This whole problem is in fact identical with the problem of the spreading of heat through a thermally conducting solid when its boundary temperature is suddenly raised from zero to some constant.

We may proceed most easily, on this occasion, by seeking a *similarity solution*. We postpone a more rational discussion of this method until §8.3; for the time being we simply observe that the equation is unchanged by the transformation of variables $y \Rightarrow \alpha y$, $t \Rightarrow \alpha^2 t$, α being a constant. This suggests the possibility that there are solutions to eqn (2.12) which are functions of y and t simply through the single combination $y/t^{\frac{1}{2}}$, for this 'similarity' variable would itself be unchanged by such a transformation. Inspection of eqn (2.12) suggests that it may be more convenient still to take $y/(\nu t)^{\frac{1}{2}}$ as the similarity variable. Thus if we try

$$u = f(\eta), \qquad \text{where } \eta = y/(\nu t)^{\frac{1}{2}}, \tag{2.13}$$

so that

$$\frac{\partial u}{\partial t} = f'(\eta)\frac{\partial \eta}{\partial t} = -f'(\eta)\frac{y}{2\nu^{\frac{1}{2}}t^{\frac{3}{2}}},$$

$$\frac{\partial u}{\partial y} = f'(\eta)\frac{\partial \eta}{\partial y} = f'(\eta)\frac{1}{\nu^{\frac{1}{2}}t^{\frac{1}{2}}}, \qquad \text{etc.,}$$

we obtain, from eqn (2.12),

$$f'' + \tfrac{1}{2}\eta f' = 0.$$

Integrating,

$$f' = Be^{-\eta^2/4},$$

whence

$$f = A + B\int_0^\eta e^{-s^2/4}\,ds,$$

where A and B are constants of integration, to be determined from the initial and boundary conditions. By virtue of eqn (2.13) these reduce to

$$f(\infty) = 0, \qquad f(0) = U,$$

so that

$$u = U\left[1 - \frac{1}{\pi^{\frac{1}{2}}}\int_0^\eta e^{-s^2/4}\,ds\right] \tag{2.14}$$

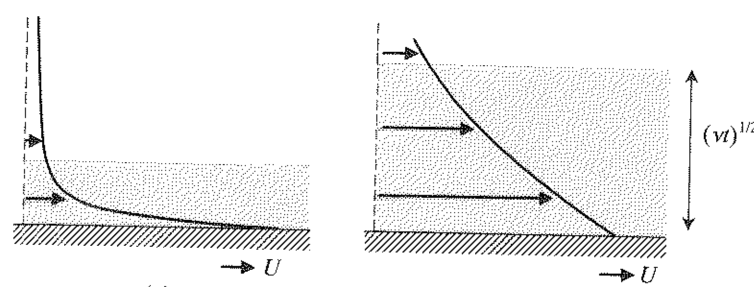
is the solution of the problem, where $\eta = y/(\nu t)^{\frac{1}{2}}$.

The simple form of the initial and boundary conditions was essential to the success of the method. The underlying reason lies in the nature of the similarity solution (2.14) itself. As its name implies, the velocity profiles $u(y)$ are, at different times, all geometrically similar. At time t_1 the velocity u is a function of $y/(\nu t_1)^{\frac{1}{2}}$; at a later time t_2 the velocity u is the *same* function of $y/(\nu t_2)^{\frac{1}{2}}$. All that happens as time goes on is that the velocity profile becomes stretched out, as indicated in Fig. 2.8. We would not expect this to be the case if, for instance, an upper boundary were present, and the solution is, indeed, not then of similarity form (see eqn (2.21)).

At time t the effects of the motion of the plane boundary are largely confined to a distance of order $(\nu t)^{\frac{1}{2}}$ from the boundary; u is less than 1% of U at $y = 4(\nu t)^{\frac{1}{2}}$. In this way viscous effects gradually communicate the motion of the boundary to the whole fluid.

A more fundamental way of viewing this process, open to considerable generalization, is in terms of the diffusion of vorticity. The vorticity is

$$\omega = -\frac{\partial u}{\partial y} = \frac{U}{(\pi \nu t)^{\frac{1}{2}}}e^{-y^2/4\nu t}, \tag{2.15}$$



(*a*) (*b*)

Fig. 2.8. The diffusion of vorticity from a plane boundary suddenly moved with velocity U. The solid line indicates the velocity profile at some early time (a) and some later time (b); the shading indicates the region of significant vorticity.

and this is exponentially small beyond a distance of order $(\nu t)^{\frac{1}{2}}$ from the boundary. The spreading of vorticity by viscous action thus smooths out what was, initially, a *vortex sheet*, i.e. an infinite concentration of vorticity at the boundary ($y=0$, $t \to 0$) with none elsewhere ($y>0$, $t \to 0$).

Finally we may state these broad conclusions in a slightly different way. Vorticity diffuses a distance of order $(\nu t)^{\frac{1}{2}}$ in time t. Equivalently, *the time taken for vorticity to diffuse a distance of order L is of the order*

$$\text{viscous diffusion time} = O(L^2/\nu). \tag{2.16}$$

Steady flow under gravity down an inclined plane

This next solution of the Navier–Stokes equations serves to make one or two elementary points about technique.

It may be argued that the key step in solving any flow problem, having decided on a sensible coordinate system, is to decide the number of independent variables (e.g. x, y, z, t) on which $\mathbf{u}$ depends, and the rule is 'the fewer, the better'.

In the present problem $\mathbf{u}$ is zero on $y=0$ (see Fig. 2.9), by virtue of the no-slip condition, so $\mathbf{u}$ must depend on y. In the absence of any a priori reason why $\mathbf{u}$ needs to depend on anything else we examine the possibility that there is a two-dimensional steady flow solution in which $\mathbf{u} = [u(y), v(y), 0]$.

Now, it is only common sense in any problem to *turn to the incompressibility condition at an early stage*, for of the two

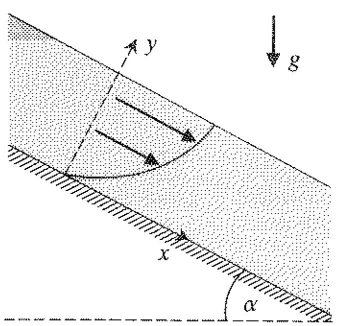


Fig. 2.9. Steady flow of a viscous fluid down an inclined plane.

equations (2.3) it is by far the simpler. In the present instance it tells us immediately that

$$dv/dy = 0,$$

i.e. that v is a constant. But $v=0$ on $y=0$, so v is zero everywhere.

Substituting $\mathbf{u} = [u(y), 0, 0]$ into the momentum equation (2.3), with the gravitational body force included, we obtain

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dy^2} + g\sin\alpha, \\ 0 &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad\qquad - g\cos\alpha. \end{aligned} \tag{2.17}$$

Integrating the second of these we find

$$p = -\rho g y \cos\alpha + f(x),$$

where $f(x)$ is an arbitrary function of x.

Now, the free surface must be $y=h$, where h is a constant, because all the streamlines are parallel to the plane. At this free surface the tangential stress must be zero and the pressure p must be equal to atmospheric pressure p_0 (see Exercise 6.3), so

$$p = p_0 \quad \text{and} \quad \mu\frac{du}{dy} = 0 \qquad \text{at } y=h, \tag{2.18}$$

by virtue of eqn (2.1). Consequently,

$$p - p_0 = \rho g(h-y)\cos\alpha,$$

whence $\partial p/\partial x$ is zero. Equation (2.17) then reduces to

$$\nu\frac{d^2u}{dy^2} = -g\sin\alpha,$$

and we may easily integrate this twice, applying the boundary conditions

$$u=0 \quad \text{at} \quad y=0, \qquad \mu\frac{du}{dy} = 0 \quad \text{at} \quad y=h,$$

to obtain

$$u = \frac{g}{2\nu}y(2h-y)\sin\alpha. \tag{2.19}$$

The velocity profile is therefore parabolic, as shown in Fig. 2.9. The volume flux down the plane, per unit length in the z-direction, is

$$Q = \int_0^h u \, dy = \frac{gh^3}{3\nu} \sin \alpha.$$

Another example of vorticity diffusion

Consider the problem in Fig. 2.10, in which a lower rigid boundary $y = 0$ is suddenly moved with speed U, while an upper rigid boundary to the fluid, $y = h$, is held at rest. As in an earlier subsection, we argue that $\boldsymbol{u} = [u(y, t), 0, 0]$ will satisfy eqn (2.12):

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{2.20}$$

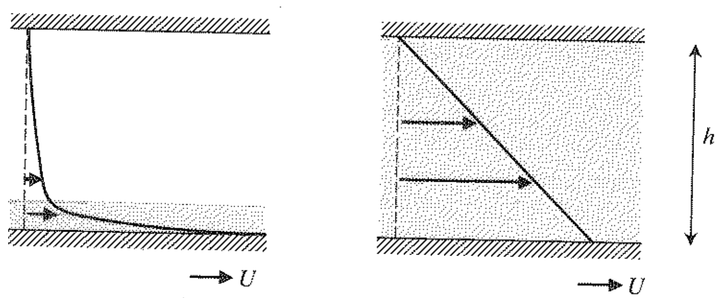
subject to the initial condition

$$u(y, 0) = 0, \qquad 0 < y < h;$$

but this time the boundary conditions will be

$$u(0, t) = U, \quad t > 0, \qquad u(h, t) = 0, \quad t > 0.$$

The equation is homogeneous, but the boundary conditions are not. Before using the method of separation of variables and



(a) $t \ll h^2/\nu$ (b) $t \gtrsim h^2/\nu$

Fig. 2.10. Flow between two rigid boundaries, one suddenly moved with speed U and one held fixed. Shading indicates regions of significant vorticity.

Fourier series we therefore reformulate the problem by first seeking a *steady* solution that satisfies the boundary conditions; this is clearly $U(1 - y/h)$. We therefore write

$$u = U(1 - y/h) + u_1,$$

where

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2},$$

$$u_1(y, 0) = -U(1 - y/h), \qquad 0 < y < h,$$

$$u_1(0, t) = 0, \qquad t > 0, \qquad u_1(h, t) = 0, \quad t > 0.$$

The boundary conditions are now homogeneous. By the method of separation of variables we find that the functions

$$\exp(-n^2\pi^2\nu t/h^2)\sin(n\pi y/h), \qquad n = 1, 2, \ldots .$$

all satisfy the equation for u_1 and the boundary conditions for u_1 at $y = 0, h$. None of these individually satisfies the initial condition for u_1, but by writing

$$u_1 = \sum_{n=1}^{\infty} A_n \exp(-n^2\pi^2\nu t/h^2)\sin(n\pi y/h),$$

we may use Fourier theory to determine the A_n such that

$$\sum_{n=1}^{\infty} A_n \sin(n\pi y/h) = -U(1 - y/h) \qquad \text{in } 0 < y < h,$$

thus satisfying the initial condition. In this way we find

$$A_n = -\frac{2}{h}\int_0^h U(1 - y/h)\sin(n\pi y/h)\, dy = -2U/n\pi,$$

and the solution is therefore

$$u(y, t) = U(1 - y/h) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp(-n^2\pi^2\nu t/h^2)\sin(n\pi y/h). \tag{2.21}$$

The main feature of this solution is that for times $t \gtrsim h^2/\nu$ (cf. eqn (2.16)) the flow has almost reached its steady state, as in Fig. 2.10(b), and the vorticity is almost distributed uniformly throughout the fluid.

2.4. Flow with circular streamlines

The Navier–Stokes equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2 \boldsymbol{u},$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

and when written out in cylindrical polar coordinates they become

$$\begin{aligned}
\frac{\partial u_r}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_r - \frac{u_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) \\
\frac{\partial u_\theta}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_\theta + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) \\
\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z,
\end{aligned} \tag{2.22}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0,$$

where

$$(\boldsymbol{u}\cdot\nabla) = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z\frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

(see eqn (A.35)).

Note the 'extra' terms that arise; the r-component of $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ is not $(\boldsymbol{u}\cdot\nabla)u_r$, for instance, but $(\boldsymbol{u}\cdot\nabla)u_r - u_\theta^2/r$ instead. This kind of thing occurs because $\boldsymbol{u} = u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta + u_z\boldsymbol{e}_z$, and *some of the unit vectors involved change with* θ:

$$\frac{\partial \boldsymbol{e}_r}{\partial \theta} = \boldsymbol{e}_\theta, \qquad \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\boldsymbol{e}_r, \qquad \frac{\partial \boldsymbol{e}_z}{\partial \theta} = 0, \tag{2.23}$$

(see eqn (A.29)). When $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ and $\nu\nabla^2\boldsymbol{u}$ are expanded carefully using these expressions they may be seen to yield eqn (2.22).

Taking explicit account of the change in direction of unit vectors may alternatively be avoided by use of the identities

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2), \tag{2.24}$$

$$\nabla^2\boldsymbol{u} = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\wedge(\nabla\wedge\boldsymbol{u}). \tag{2.25}$$

For this purpose we recall

$$\nabla \wedge \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{vmatrix} \tag{2.26}$$

(see Exercise 2.13).

The differential equation for circular flow

Consider solutions to the Navier–Stokes equations of the form

$$\boldsymbol{u} = u_\theta(r, t)\boldsymbol{e}_\theta, \tag{2.27}$$

so that the streamlines are circular. The incompressibility condition $\nabla\cdot\boldsymbol{u} = 0$ is automatically satisfied for any flow of the form (2.27).

Rather than use the remaining equations in the ready-made form (2.22) it is instructive to derive them, for the flow (2.27), using the expressions (2.23). Thus

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{u_\theta}{r}\frac{\partial}{\partial \theta}[u_\theta(r,t)\boldsymbol{e}_\theta] = \frac{u_\theta^2}{r}\frac{\partial \boldsymbol{e}_\theta}{\partial \theta} = -\frac{u_\theta^2}{r}\boldsymbol{e}_r, \tag{2.28}$$

while

$$\nu\nabla^2\boldsymbol{u} = \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)[u_\theta(r,t)\boldsymbol{e}_\theta],$$

and

$$\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}[u_\theta\boldsymbol{e}_\theta] = \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(u_\theta\frac{\partial \boldsymbol{e}_\theta}{\partial \theta}\right) = \frac{-1}{r^2}\frac{\partial}{\partial \theta}(u_\theta\boldsymbol{e}_r) = -\frac{u_\theta}{r^2}\boldsymbol{e}_\theta,$$

so

$$\nu\nabla^2\boldsymbol{u} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)\boldsymbol{e}_\theta. \tag{2.29}$$

When $\boldsymbol{u} = u_\theta(r, t)\boldsymbol{e}_\theta$ the Navier–Stokes equations therefore reduce to

$$-\frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r},$$

$$\frac{\partial u_\theta}{\partial t} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right),$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

as we might have deduced more quickly from eqn (2.22).

Now, u_θ is a function of r and t only, so from the second equation the same must be true of $\partial p/\partial\theta$, so $\partial p/\partial\theta = P(r, t)$, say. Integrating:

$$p = P(r, t)\theta + f(r, t),$$

as $\partial p/\partial z = 0$. We conclude that $P(r, t) = 0$, for otherwise p would be a multivalued function of position (different at $\theta = 0$ and at $\theta = 2\pi$, say). Thus

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right) \tag{2.30}$$

is the evolution equation for a viscous flow with $\boldsymbol{u} = u_\theta(r, t)\boldsymbol{e}_\theta$.

Steady flow between rotating cylinders

For steady flow we have

$$r^2\frac{d^2u_\theta}{dr^2} + r\frac{du_\theta}{dr} - u_\theta = 0,$$

with general solution

$$u_\theta = Ar + \frac{B}{r}. \tag{2.31}$$

If the fluid occupies the gap $r_1 \leqslant r \leqslant r_2$ between two circular cylinders which rotate with angular velocities Ω_1 and Ω_2, then we may apply the no-slip condition at each cylinder to obtain

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \qquad B = \frac{(\Omega_1 - \Omega_2)r_1^2 r_2^2}{r_2^2 - r_1^2}. \tag{2.32}$$

The most interesting thing about this flow is the manner in which it becomes unstable if Ω_1 is too large, so that superbly regular and axisymmetric *Taylor vortices* appear (see §9.4, especially Fig. 9.8).

Spin-down in an infinitely long circular cylinder

Suppose viscous fluid occupies the region $r \leqslant a$ within a circular cylinder of radius a, and suppose that both cylinder and fluid are initially rotating with uniform angular velocity Ω, so that

$$u_\theta = \Omega r, \qquad r \leqslant a, \qquad t = 0.$$

Suppose that the cylinder is then suddenly brought to rest. We need to solve

$$\frac{\partial u_\theta}{\partial t} = \nu\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2}\right)$$

with the above initial condition and the boundary condition

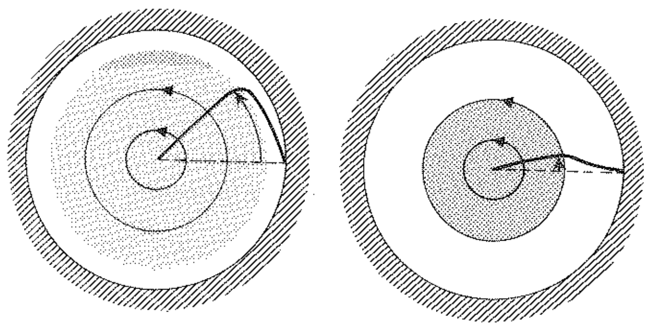
$$u_\theta = 0 \qquad \text{at } r = a, \quad t > 0.$$

The problem may be tackled in a Fourier-series type manner, as for eqn (2.21), but the separable solutions now involve Bessel functions, and

$$u_\theta(r, t) = -2\Omega a \sum_{n=1}^{\infty} \frac{\mathrm{J}_1(\lambda_n r/a)}{\lambda_n \mathrm{J}_0(\lambda_n)} \exp\left(-\lambda_n^2 \frac{\nu t}{a^2}\right). \tag{2.33}$$

Here λ_n denote the positive values of λ at which $\mathrm{J}_1(\lambda) = 0$, and J_k denotes the Bessel function of order k. All the terms of the series decay rapidly with t; the one that survives longest is the first one, and $\lambda_1 \doteqdot 3.83$. The 'spin-down' process is therefore well under way in a time of order $a^2/\nu\lambda_1^2$, i.e. in the classic viscous diffusion time (2.16).

If we apply this to a stirred cup of tea, with $a = 4$ cm and $\nu = 10^{-2}\,\mathrm{cm^2\,s^{-1}}$ for water, we obtain a 'spin-down' time of about 2 minutes. This is much too long; casual observation suggests that u_θ drops to about $1/e$ of its original value in about 15 s. The discrepancy arises because straightforward diffusion of (negative) vorticity from the side walls is not the key process by which a stirred cup of tea comes to rest; the bottom of the cup—wholly absent in the present model—plays a crucial role (see Fig. 5.6.)



(a) (b)

Fig. 2.11. 'Spin-down' in an infinitely long circular cylinder. Initially there is vorticity 2Ω everywhere, but negative vorticity diffuses inward from the stationary boundary $r = a$, so that the (shaded) region of significant vorticity shrinks with time.

Viscous decay of a line vortex

The line vortex

$$\boldsymbol{u} = \frac{\Gamma_0}{2\pi r}\boldsymbol{e}_\theta, \tag{2.34}$$

where Γ_0 is a constant, has zero vorticity in $r > 0$ but infinite vorticity at $r = 0$. In a viscous fluid, then, this flow does not persist; the vorticity diffuses outward as time goes on.

To examine this process it is convenient to take the circulation

$$\Gamma(r, t) = 2\pi r u_\theta(r, t) \tag{2.35}$$

as the dependent variable of the problem. In place of eqn (2.30) we then obtain

$$\frac{\partial \Gamma}{\partial t} = \nu\left(\frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r}\frac{\partial \Gamma}{\partial r}\right). \tag{2.36}$$

The initial condition is

$$\Gamma(r, 0) = \Gamma_0.$$

We require u_θ finite at $r = 0$ at any later time, so

$$\Gamma(0, t) = 0, \qquad t > 0.$$

This problem is very similar to that in which a plane rigid boundary is jerked into motion (see eqn (2.14)); we leave it as an exercise to seek, as in that case, a similarity solution in which

$$\Gamma = f(\eta), \qquad \text{where } \eta = r/(\nu t)^{\frac{1}{2}}.$$

In this way we may discover that

$$\Gamma = \Gamma_0(1 - e^{-r^2/4\nu t}),$$

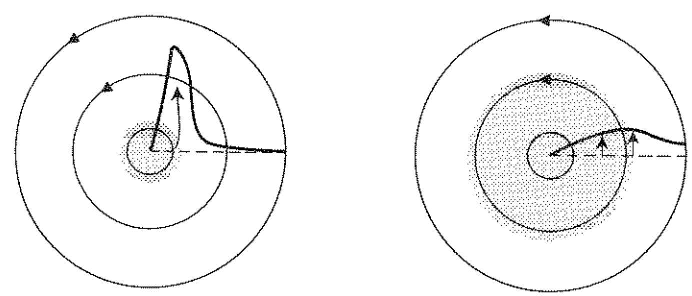
so

$$u_\theta = \frac{\Gamma_0}{2\pi r}(1 - e^{-r^2/4\nu t}). \tag{2.37}$$

At distances greater than about $(4\nu t)^{\frac{1}{2}}$ from the axis the circulation is almost unaltered, because very little vorticity has yet diffused that far out. At small distances from the axis, however, where $r \ll (4\nu t)^{\frac{1}{2}}$, the flow is no longer remotely irrotational; indeed

$$u_\theta \doteqdot \frac{\Gamma_0}{8\pi\nu t} r \qquad \text{for} \qquad r \ll (4\nu t)^{\frac{1}{2}}, \tag{2.38}$$

which corresponds to almost uniform rotation with angular velocity $\Gamma_0/8\pi\nu t$. The intensity of the vortex thus decreases with time as the 'core' spreads radially outward (Fig. 2.12).



(a) (b)

Fig. 2.12. The viscous diffusion of a vortex.

2.5. The convection and diffusion of vorticity

If we take the curl of the momentum equation (2.3) we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \nu\, \nabla^2 \boldsymbol{\omega}, \tag{2.39}$$

(cf. eqn (1.25)), and in the case of a 2-D flow this reduces to

$$\frac{\partial \omega}{\partial t} + (\boldsymbol{u} \cdot \nabla)\omega = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right). \tag{2.40}$$

In Chapter 1 we set the viscosity ν to zero from the outset; ω was then conserved by individual fluid elements in 2-D flow. Changes in ω at a particular point in space took place only by the *convection* of vorticity from elsewhere in the fluid, and this process is represented by the second term in eqn (2.40). In §§2.3 and 2.4, on the other hand, we looked at some simple viscous flow problems in which the term $(\boldsymbol{u} \cdot \nabla)\omega$ happened to be identically zero; in other words, we isolated *diffusion* of vorticity as a mechanism, this being represented by the third term in eqn (2.40).

In general, there is both diffusion and convection of vorticity in a viscous fluid flow, and we end this chapter with two examples.

2-D flow near a stagnation point

The main features of this exact solution of the Navier–Stokes equations (Exercise 2.14) are as follows. First, there is an inviscid 'mainstream' flow

$$u = \alpha x, \qquad v = -\alpha y, \tag{2.41}$$

where α is a positive constant. This fails to satisfy the no-slip condition at the rigid boundary $y = 0$, but the mainstream flow speed $\alpha|x|$ increases with distance $|x|$ along the boundary. By Bernoulli's theorem, the mainstream pressure p *decreases* with distance along the boundary in the flow direction (Fig. 2.13), so we may hope for a thin, unseparated boundary layer which adjusts the velocity to satisfy the no-slip condition (see §2.1). This is indeed the case, as Exercise 2.14 shows, and the boundary layer, in which all the vorticity is concentrated, has thickness

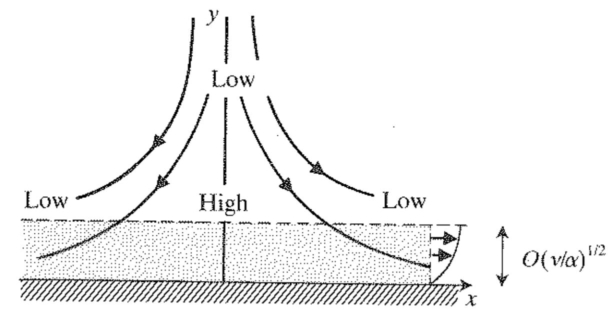


Fig. 2.13. Flow towards a 2-D stagnation point.

$\delta = O(\nu/\alpha)^{\frac{1}{2}}$. In this boundary layer there is a steady state balance between the viscous diffusion of vorticity from the wall and the convection of vorticity towards the wall by the flow. Thus if ν decreases the diffusive effect is weakened, while if α increases the convective effect is enhanced; in either case the boundary layer becomes thinner.

High Reynolds number flow past a flat plate

In uniform flow past a flat plate with a leading edge, as in Fig. 2.14, there is no flow component convecting vorticity towards the plate to counter the diffusion of vorticity from it, so the boundary layer becomes progressively thicker with downstream distance x. (In less formal terms, the layers of fluid closest to the centreline are the first to be slowed down as they pass the leading edge, and they in turn gradually slow down the layers of fluid which are further away.)

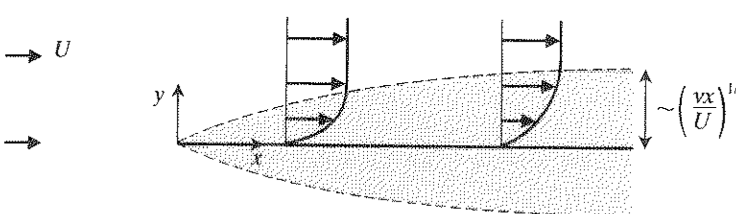


Fig. 2.14. The boundary layer on a flat plate.

We may estimate the boundary layer thickness δ by a simpl argument based on the related problem in which the flat plate i instead suddenly pulled to the left, with speed U, through flui which was previously at rest. From Fig. 2.8 we infer that at time after the plate is moved vorticity will have diffused out a distanc of order $(\nu t)^{\frac{1}{2}}$. But by this time the leading edge of the plate wi have moved a distance $x = Ut$ to the left. It follows that a distance x downstream from the leading edge there will b significant vorticity a distance of order

$$\delta \sim (\nu x/U)^{\frac{1}{2}} \qquad (2.42$$

from the plate, but not beyond.

This crude estimate for the growth of the boundary layer wit downstream distance x in Fig. 2.14 is indeed confirmed by th appropriate solution of the boundary layer equations (see §8.3). For a plate of finite length L the thickness (2.42) is in keepin with the claim (2.7) and is small compared with L at all points o the plate if $R = UL/\nu \gg 1$.

Exercises

2.1. Give an order of magnitude estimate of the Reynolds number for:

(i) flow past the wing of a jumbo jet at $150\,\mathrm{m\,s^{-1}}$ (roughly half the speed of sound);

(ii) the experiment in §1.1 with, say, $L = 2$ cm and $U = 5\,\mathrm{cm\,s^{-1}}$;

(iii) a thick layer of golden syrup draining off a spoon;

(iv) a spermatozoan with tail length of 10^{-3} cm swimming at $10^{-2}\,\mathrm{cm\,s^{-1}}$ in water.

Give an order of magnitude estimate of the thickness of the boundary layer in case (i).

2.2. The problem of 2-D steady viscous flow past a circular cylinder of radius a involves finding a velocity field $\boldsymbol{u} = [u(x, y), v(x, y), 0]$ which satisfies

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0,$$

together with the boundary conditions

$$\boldsymbol{u} = 0 \quad \text{on } x^2 + y^2 = a^2; \qquad \boldsymbol{u} \to (U, 0, 0) \quad \text{as } x^2 + y^2 \to \infty.$$

Rewrite this problem in *dimensionless form* by using the dimensionless variables

$$\boldsymbol{x}' = \boldsymbol{x}/a, \qquad \boldsymbol{u}' = \boldsymbol{u}/U, \qquad p' = p/\rho U^2$$

in places of $\boldsymbol{x}$, $\boldsymbol{u}$, and p. Without attempting to solve the problem, show that the streamline pattern can depend on ν, a, and U only in the combination $R = Ua/\nu$, so that flows at equal Reynolds numbers are geometrically similar.

2.3. (i) Viscous fluid flows between two stationary rigid boundaries $y = \pm h$ under a constant pressure gradient $P = -\mathrm{d}p/\mathrm{d}x$. Show that

$$u = \frac{P}{2\mu}(h^2 - y^2), \qquad v = w = 0.$$

(ii) Viscous fluid flows down a pipe of circular cross-section $r = a$ under a constant pressure gradient $P = -\mathrm{d}p/\mathrm{d}z$. Show that

$$u_z = \frac{P}{4\mu}(a^2 - r^2), \qquad u_r = u_\theta = 0.$$

[These are called *Poiseuille flows* (Fig. 2.15), after the physician who first studied (ii) in connection with blood flow. Their instability at high Reynolds number constitutes one of the most important problems of fluid dynamics (see §9.1).]

2.4. Two incompressible viscous fluids of the same density ρ flow, one on top of the other, down an inclined plane making an angle α with the horizontal. Their viscosities are μ_1 and μ_2, the lower fluid is of depth h_1 and the upper fluid is of depth h_2. Show that

$$u_1(y) = [(h_1 + h_2)y - \tfrac{1}{2}y^2]\frac{g \sin \alpha}{\nu_1},$$

so that the velocity of the lower fluid $u_1(y)$ is dependent on the depth h_2, but not the viscosity, of the upper fluid. Why is this?

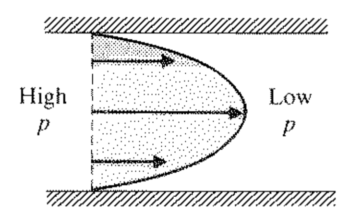


Fig. 2.15. Poiseuille flow.

2.5. Viscous fluid is at rest in a two-dimensional channel between tw stationary rigid walls $y = \pm h$. For $t \geqslant 0$ a constant pressure gradien $P = -\mathrm{d}p/\mathrm{d}x$ is imposed. Show that $u(y, t)$ satisfies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \frac{P}{\rho},$$

and give suitable initial and boundary conditions. Find $u(y, t)$ in th form of a Fourier series, and show that the flow approximates to stead channel flow when $t \gg h^2/\nu$.

2.6. Viscous fluid flows between two rigid boundaries $y = 0$, $y = h$, th lower boundary moving in the x-direction with constant speed U, th upper boundary being at rest. The boundaries are porous, and th vertical velocity v is $-v_0$ at each one, v_0 being a given constant (so tha there is an imposed flow across the system). Show that the resulting flov is

$$u = U\left(\frac{e^{-v_0 y/\nu} - e^{-v_0 h/\nu}}{1 - e^{-v_0 h/\nu}}\right), \qquad v = -v_0.$$

Show that the horizontal velocity profile $u(y)$ is as in Fig. 2.16, so tha when $v_0 h/\nu$ is large the downflow v_0 confines the vorticity to a very thi layer adjacent to $y = 0$.

[This is probably the mathematically simplest example of a stead boundary layer, but it is untypical in that the boundary layer thickness i proportional to ν, rather than to $\nu^{\frac{1}{2}}$ (see eqn (2.7)).]

2.7. Incompressible fluid occupies the space $0 < y < \infty$ above a plane rigid boundary $y = 0$ which oscillates to and fro in the x-direction with velocity $U \cos \omega t$. Show that the velocity field $\boldsymbol{u} = [u(y, t), 0, 0]$ satisfies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

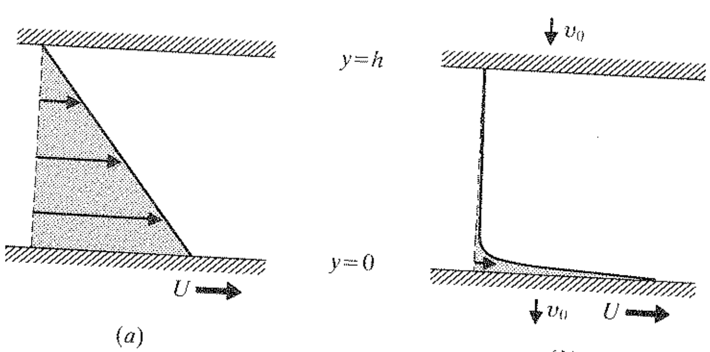


Fig. 2.16. Wall-driven channel flow with (a) $v_0 = 0$ and (b) $v_0 h/\nu \gg 1$.

(there being no applied pressure gradient), and by seeking a solution of the form

$$u = \mathscr{R}[f(y)e^{i\omega t}],$$

where $\mathscr{R}$ denotes 'real part of', show that

$$u(y, t) = Ue^{-ky}\cos(ky - \omega t),$$

where $k = (\omega/2\nu)^{\frac{1}{2}}$.

Sketch the velocity profile at some time t, and note that there is hardly any motion beyond a distance of order $(\nu/\omega)^{\frac{1}{2}}$ from the boundary.

2.8. A circular cylinder of radius a rotates with constant angular velocity Ω in a viscous fluid. Show that the line vortex flow

$$\boldsymbol{u} = \frac{\Omega a^2}{r}\boldsymbol{e}_\theta, \qquad r \geqslant a,$$

is an exact solution of the equations and boundary conditions. Describe roughly how the vorticity changes with time when the cylinder is suddenly started into rotation with angular velocity Ω from a state of rest. Likewise, discuss the case in which an outer cylinder $r = b$ is simultaneously given an angular velocity $\Omega a^2/b^2$.

2.9. A viscous flow is generated in $r \geqslant a$ by a circular cylinder $r = a$ which rotates with constant angular velocity Ω. There is also a radial inflow which results from a uniform suction on the (porous) cylinder, so that $u_r = -U$ on $r = a$. Show that

$$u_r = -Ua/r \qquad \text{for } r \geqslant a,$$

and that

$$r^2\frac{\mathrm{d}^2 u_\theta}{\mathrm{d}r^2} + (R+1)r\frac{\mathrm{d}u_\theta}{\mathrm{d}r} + (R-1)u_\theta = 0,$$

where $R = Ua/\nu$.

Show that if $R < 2$ there is just one solution of this equation which satisfies the no-slip condition on $r = a$ and has finite circulation $\Gamma = 2\pi r u_\theta$ at infinity, but that if $R > 2$ there are many such solutions.

2.10. Show that, as claimed in eqn (2.37), a line vortex of strength Γ_0 decays by viscous diffusion in the following manner:

$$u_\theta = \frac{\Gamma_0}{2\pi r}(1 - e^{-r^2/4\nu t}).$$

Calculate and sketch the vorticity as a function of r at two different times.

2.11. Viscous fluid occupies the region $0<z<h$ between two rigi boundaries $z=0$ and $z=h$. The lower boundary is at rest, the uppe boundary rotates with constant angular velocity Ω about the z-axis Show that a steady solution of the full Navier–Stokes equations of th form

$$\boldsymbol{u}=u_\theta(r,z)\boldsymbol{e}_\theta$$

is not possible, so that any rotary motion $u_\theta(r,z)$ in this system must b accompanied by a *secondary flow* $(u_r, u_z \neq 0)$.

2.12. Viscous fluid is inside an infinitely long circular cylinder $r=a$ which is rotating with angular velocity Ω, so that $u_\theta=\Omega r$ for $r\leqslant a$. The cylinder is suddenly brought to rest at $t=0$. Rewrite the evolution equation (2.30) in the form

$$\frac{\partial u_\theta}{\partial t}=\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{\nu u_\theta}{r^2},$$

and thereby show that

$$\frac{\mathrm{d}E}{\mathrm{d}t}+\frac{2\nu}{a^2}E\leqslant 0,$$

where

$$E=\tfrac{1}{2}\int_0^a r u_\theta^2\,\mathrm{d}r,$$

which is proportional to the kinetic energy of the flow. Hence show that $E\to 0$ as $t\to\infty$.

[This may seem a little pointless, given that the exact solution (2.33) is available, but the above approach is in fact of very general value, and provides the basis for the proof, in §9.7, of an important uniqueness theorem.]

2.13. Re-derive the results (2.28) and (2.29) by the alternative route involving eqns (2.24), (2.25), and (2.26).

2.14. Consider in $y\geqslant 0$ the 2-D flow

$$u=\alpha x f'(\eta),\qquad v=-(\nu\alpha)^{\frac{1}{2}}f(\eta),$$

where

$$\eta=(\alpha/\nu)^{\frac{1}{2}}y.$$

Show that it is an exact solution of the Navier–Stokes equations which (i) satisfies the boundary conditions at the stationary rigid boundary $y=0$ and (ii) takes the asymptotic form $u\sim\alpha x$, $v\sim-\alpha y$ far from the

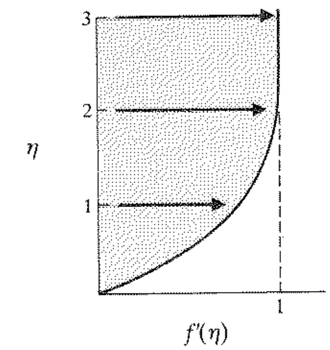


Fig. 2.17. The velocity profile in the boundary layer near a 2-D stagnation point.

boundary (see Fig. 2.13) if

$$f'''+ff''+1-f'^2=0,$$

with

$$f(0)=f'(0)=0,\qquad f'(\infty)=1.$$

[The differential equation for $f(\eta)$ is solved numerically, and $f'(\eta)$ is shown in Fig. 2.17. Notably, $f'(3)=0.998$, so beyond a distance of $3(\nu/\alpha)^{\frac{1}{2}}$ from the boundary the flow is effectively inviscid and irrotational, with $u\doteqdot\alpha x$ and $v\doteqdot-\alpha y$.]

2.15. If a flat plate is fixed between $(0,0)$ and $(0,L)$ in Fig. 2.13, with $L\gg(\nu/\alpha)^{\frac{1}{2}}$, one might at first think that the flow would not be much affected, for the plate lies along one of the streamlines of the original flow. Why is it, then, that the observed flow is quite different, as in Fig. 2.18?

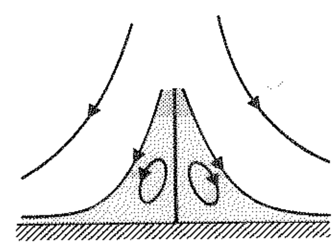


Fig. 2.18. High Reynolds number stagnation-point flow with a protruding flat plate.