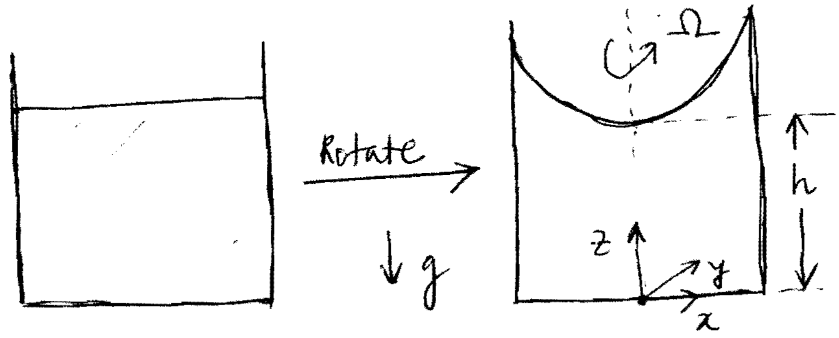
BAIDURJA RAY
4/13/11

MAE 4230/5230
SOLUTION TO HOMEWORK 8

⟨2⟩ Rotating bucket of water



An ideal fluid is rotating under gravity g with constant angular velocity Ω.
From Acheson Pg 5, the velocity field is given by $\underline{u} = (-\Omega y, \Omega x, 0)$
Let us now solve the Euler equations to determine the pressure field $P(x,y,z)$ and hence the shape of the free surface (where $P = P_0$)

$$X: \quad \cancel{\frac{\partial u}{\partial t}}^{\,0} + u\cancel{\frac{\partial u}{\partial x}}^{\,0} + v\frac{\partial u}{\partial y} + \cancel{w}^{\,0}\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad \text{—(1)}$$

$$\Rightarrow (\Omega x)(-\Omega) = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$\Rightarrow \frac{\partial P}{\partial x} = \rho\Omega^2 x$$

$$\Rightarrow P(x,y,z) = \frac{\rho\Omega^2 x^2}{2} + C_1(y,z) \quad \text{—(A)}$$

$$y: \quad \cancel{\frac{\partial v}{\partial t}} + u\frac{\partial v}{\partial x} + v\cancel{\frac{\partial v}{\partial y}} + \cancel{w}\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad \text{— (2)}$$

$$\Rightarrow (-\Omega y)(-\Omega) = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \rho\Omega^2 y$$

$$\Rightarrow P(x,y,z) = \frac{\rho\Omega^2 y^2}{2} + C_2(x,z) \quad \text{— (B)}$$

$$z: \quad \cancel{\frac{\partial w}{\partial t}} + u\cancel{\frac{\partial w}{\partial x}} + v\cancel{\frac{\partial w}{\partial y}} + w\cancel{\frac{\partial w}{\partial z}} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g \quad \text{— (3)}$$

$$\Rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial z} = g$$

$$\Rightarrow P(x,y,z) = -\rho g z + C_3(x,y) \quad \text{— (C)}$$

Now, the pressure field $P(x,y,z)$ satisfies equations (A), (B) and (C) simultaneously.
This is possible only if

$$P(x,y,z) = \frac{\rho\Omega^2 x^2}{2} + \frac{\rho\Omega^2 y^2}{2} - \rho g z + K$$

where K = constant
At a point on the surface, say $(0,0,h)$,
$P = P_0$.

$$\therefore P_0 = -\rho g h + K$$

$$\Rightarrow K = P_0 + \rho g h$$

$$\therefore \quad \boxed{P(x,y,z) = \frac{\rho}{2}\Omega^2(x^2+y^2) - \rho g(z-h) + P_0}$$

Now, in order to find the shape of the free surface i.e., the locus of the free surface, we use the condition that at any point (x,y,z) <u>on the free surface</u>, $P = P_0$.

$$\therefore \quad \cancel{P_0} = \frac{\rho}{2}\Omega^2(x^2+y^2) - \rho g(z-h) + \cancel{P_0}$$

$$\Rightarrow \quad \boxed{z = \frac{\Omega^2}{2g}(x^2+y^2) + h}$$

We can easily verify that this is the equation of a paraboloid. Also,

$\frac{\partial z}{\partial x} = 0$ @ $x = 0$ and $\frac{\partial z}{\partial y} = 0$ @ $y = 0$

with $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2} > 0$

$\therefore$ $(x=0, y=0)$ represents a minima and the water level is lowest in the middle.

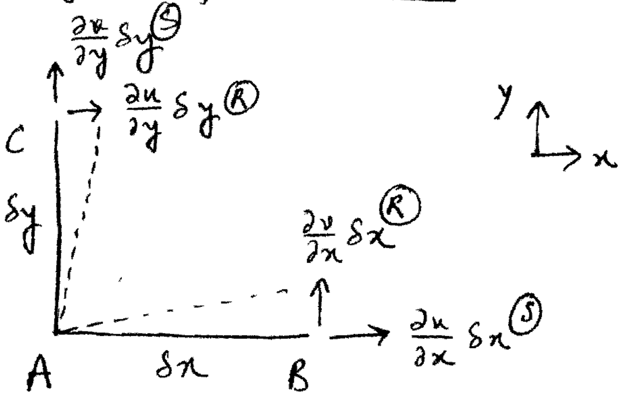
<u>Bernoulli's equation</u> can be applied in a rotational flow only along streamlines. In this case, the streamlines are circular due to rotation and the free surface cuts across the streamlines. Therefore, Bernoulli's

equation cannot be used to determine the equation of the free surface.

⟨3⟩ A fluid is undergoing solid body rotation Ω. We have to show that the vorticity is $\omega = 2\Omega$ in three ways:

(a) <u>Refer to Section 1.4, Pg 11 of Acheson</u>

Consider two short mutually ⊥ line elements AB and AC emanating from a fluid particle A.



We can observe that the fluid lines will stretch due to the components marked as (S) and <u>rotate</u> due to the components marked as (R).

<u>Vorticity ω</u> at a point in the flow field is defined as <u>twice</u> the average angular velocity of two ⊥ fluid lines at that point. Therefore, assuming CCW rotation as positive, and recognizing that $\frac{\partial v}{\partial x}$ and $\frac{\partial u}{\partial y}$ represent the instantaneous

angular velocities of lines AB and AC respectively, we get,

$$\omega = 2\,\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

from the previous problem, we know,

$$\underline{u} = (-\Omega y, \Omega x, 0)$$

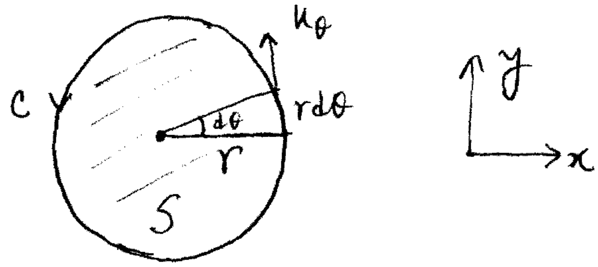
$$\therefore \quad \omega = (\Omega - (-\Omega)) \Rightarrow \boxed{\omega = 2\Omega}$$

(b) Circulation Γ around a closed contour C in the flow field is given by

$$\Gamma = \oint_C \underline{u}\,.\,\underline{dl}$$

By Stokes' theorem,

$$\Gamma = \oint_C \underline{u}\,.\,\underline{dl} = \int_S (\nabla \times \underline{u})\,.\,\hat{n}\,dS \quad \text{—— (1)}$$



Consider a circular closed contour in the x-y plane. The velocity field in r-θ-z coordinates can be written as

$$\underline{u} = 0\,\hat{e_r} + \underline{\underline{\Omega r}}\,\hat{e_\theta} + 0\,\hat{e_z}$$

$\therefore$ Equation (1) becomes,

$$\Gamma = \oint_C \underline{u} \cdot \underline{dl} = \int_S (\underline{\nabla} \times \underline{u}) \cdot \hat{n} \, dS$$

$$\Rightarrow \quad \int_S \underline{\omega} \cdot \hat{n} \, dS = \oint_C u_\theta \, r \, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \Omega r \, r \, d\theta$$

$$= \int_0^{2\pi} \Omega r^2 \, d\theta$$

$$= \Omega r^2 (2\pi)$$

Since the angular velocity Ω is a constant, the vorticity is also a constant.

$$\therefore \quad \int_S \underline{\omega} \cdot \hat{n} \, dS = \omega \int_S dS = \pi r^2 \omega$$

$$\therefore \quad \cancel{\pi} \cancel{r^2} \omega = \Omega \cancel{r^2} \, 2 \cancel{\pi}$$

$$\Rightarrow \quad \boxed{\omega = 2\Omega}$$

(c) In cartesian coordinates,

$$\underline{\omega} = \underline{\nabla} \times \underline{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\Omega y & \Omega x & 0 \end{vmatrix} = \hat{i}(0) + \hat{j}(0) + \hat{k}(2\Omega)$$

$$\Rightarrow \quad \boxed{\underline{\omega} = 2\Omega \hat{k}}$$

⟨4⟩ Vorticity of flow around a line vortex

Velocity field generated by a vortex line, $u_\theta = 1/r$. Let us apply the curl in cylindrical coordinates.

$$(\underline{\nabla} \times \underline{V})_r = \omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} = 0$$

$$(\underline{\nabla} \times \underline{V})_\theta = \omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0$$

$$(\underline{\nabla} \times \underline{V})_z = \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \cancel{\frac{1}{r}\frac{\partial u_r}{\partial \theta}} \quad \text{no}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(1) = 0 \quad (r \neq 0)$$

$$\therefore \boxed{\underline{\omega} = 0}$$

However, the <u>circulation</u> around a circular contour C of radius r,

$$\Gamma = \oint_C \underline{u} \cdot \underline{dl} = \int_0^{2\pi} \frac{1}{\cancel{r}} \cancel{r}\, d\theta = 2\pi \neq 0$$

So, vorticity $\omega = 0$ everywhere except $r=0$ while circulation around any contour is constant. Hence, <u>all the vorticity in the flow must be concentrated at $r=0$</u>.

<5> Vorticity of a rotating Couette flow

The flow inside a rotating Couette cell is given by $u_\theta(r) = Ar + B/r$.

(a) From answers to questions <3> and <4> the vorticity of this flow is

$$\boxed{\omega = 2A}$$

(b) We can show that $A = \dfrac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}$

Clearly A has dimensions of angular velocity. If the outer cylinder were fixed, i.e., $\Omega_2 = 0$ then

$$A = \frac{-\Omega_1 r_1^2}{r_2^2 - r_1^2}$$

If the inner cylinder were fixed, i.e., $\Omega_1 = 0$ then

$$A = \frac{\Omega_2 r_2^2}{r_2^2 - r_1^2}$$

Therefore, A represents the difference between the vorticity of the solid body rotations of the outer and the inner cylinder.

⟨6⟩ Streamlines

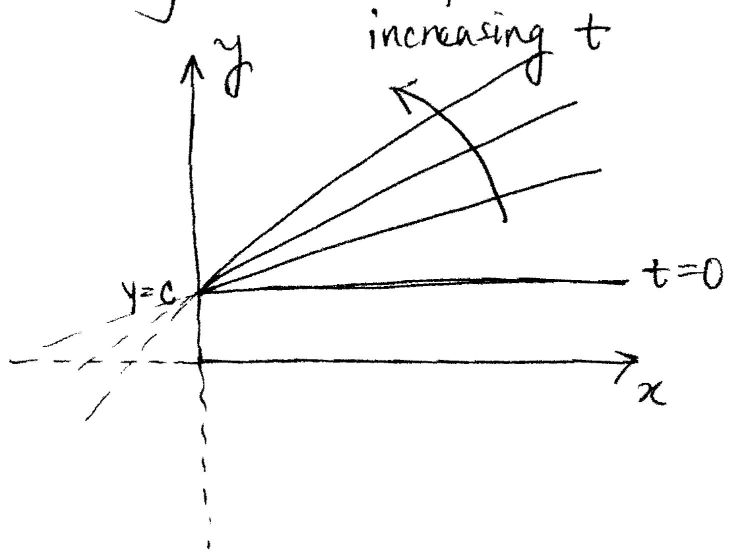
We have an unsteady 2D flow, $u = u_0$, $v = kt$; k and u_0 are positive constants.

Equation of the streamlines

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dy}{dx} = \frac{v}{u} = \frac{kt}{u_0}$$

$$\Rightarrow \boxed{y = \left(\frac{kt}{u_0}\right)x + c}$$

Hence, at each instant of time, the streamlines are straight lines with the slope being time-dependent.



For any fluid particle, $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$ with initial conditions (x_0, y_0) at $t = 0$

$$\therefore \frac{dx}{dt} = u_0 \Rightarrow x = u_0 t + C_1$$

$$\Rightarrow \underline{(x - x_0) = u_0 t} \quad — (1)$$

$\therefore \frac{dy}{dt} = v \Rightarrow \frac{dy}{dt} = kt$

$\Rightarrow y = kt^2/2 + C_2$

$\Rightarrow (y - y_0) = \frac{kt^2}{2}$ —— (2)

From (1) and (2),

$(x - x_0)^2 \cdot \frac{k}{2u_0^2} = (y - y_0)$

$\Rightarrow \boxed{(y - y_0) = \left(\frac{k}{2u_0^2}\right)(x - x_0)^2}$

This is the locus of the path followed by a fluid particle. Clearly, it is parabolic with focal length $\left(\frac{u_0^2}{2k}\right)$.

$(x - x_0)^2 = \left(\frac{2u_0^2}{k}\right)(y - y_0)$

$\Rightarrow (x - x_0)^2 = 4\left(\frac{u_0^2}{2k}\right)(y - y_0)$

⟨7⟩ Vector Potential and Stream Function

Vector potential ϕ : $\quad u = \frac{\partial \phi}{\partial x}$, $\quad v = \frac{\partial \phi}{\partial y}$

Continuity for incompressible flow

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$ — Laplace Equation

Stream Function ψ : $\quad u = +\frac{\partial \psi}{\partial y}$, $\quad v = -\frac{\partial \psi}{\partial x}$

For an irrotational flow,

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\Rightarrow \quad -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \quad \boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0}$$ — Laplace equation

Hence, for an incompressible, irrotational and 2D flow, both velocity potential and stream function satisfy Laplace equation.