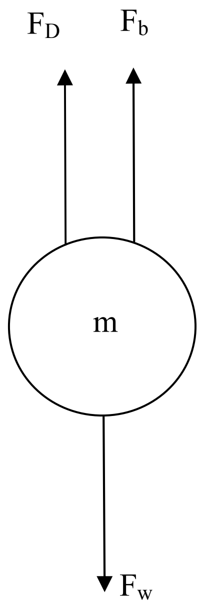
Homework 2 Solutions

2.

At terminal velocity, the free-body diagram for the sphere is given as:

Free-Body Diagram



Where F_b is the buoyant force, F_D is the Stokes drag force, and F_W is the weight of the sphere. At terminal velocity, the sphere is not accelerating, thus Newton's 2nd law reduces to:

$$F_D + F_b = F_w$$

The general form of this equation is:

$$6\pi\mu UR + \rho_{fluid} gV = mg$$

Or alternatively:

$$6\pi\mu UR + \rho_{fluid} \cdot g \cdot 4/3 \cdot \pi \cdot R^3 = \rho_{sphere} \cdot 4/3 \cdot \pi \cdot R^3 \cdot g$$

Solving for U:

$$U = \frac{2}{9} \cdot \frac{g}{\mu} \cdot R^2 \cdot (\rho_{sphere} - \rho_{fluid})$$

Note this formula for setting velocity is valid only for low Reynolds number flow.

3.

a)

The Navier-Stokes equation for incompressible flow of Newtonian-fluids is given by:

$$\rho\left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla\vec{u}\right] = -\nabla p + \mu\nabla^2\vec{u} \quad (3.1)$$

The steady and fully developed assumptions allow us to neglect both terms on the left-hand side of this equation. Writing this equation for uni-directional flow in cylindrical coordinates, 3.1 further reduces to:

$$\mu \cdot \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{dp}{dz} \quad (3.2)$$

Separating, and integrating twice, we obtain the general solution for steady, fully-developed, uni-directional flow in a pipe:

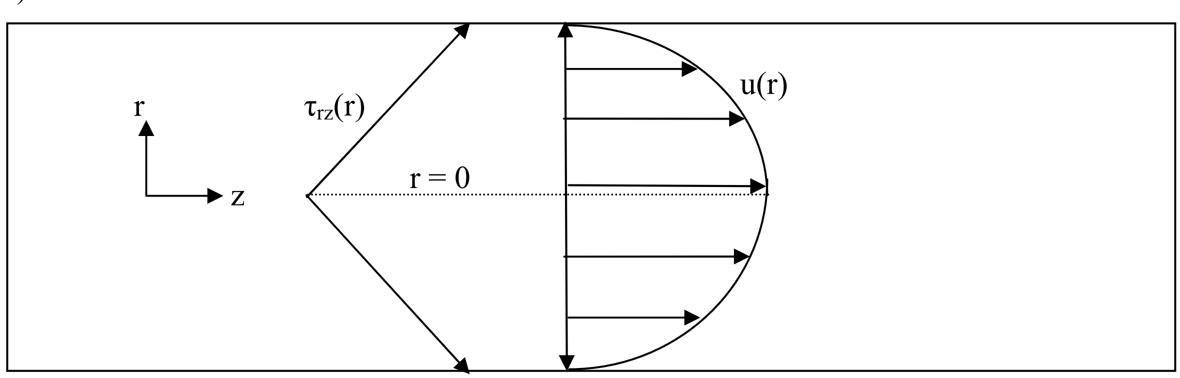
$$u(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + C_1 lnr + C_2 \quad (3.3)$$

Notice the curious logarithmic term. The Navier-Stokes equation along with our assumptions predict an un-bounded velocity as r goes to zero. This result however is not physical for flow in a pipe, much less a real flow. We employ the condition that the velocity must remain bounded for all values of r. Thus, C_1 must be zero, and 3.3 reduces to:

$$u(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + C_2 \quad (3.4)$$

We could employ the no-slip condition to gain exact solution for the flow, but we only asked for the general solution. 3.4 is physical enough to draw the qualitative picture.

b)



The velocity and shear stress profiles are plotted above. We note that the fluid flows to the right in this case, indicating that the pressure gradient is negative. The maximum velocity occurs at r = 0, and the velocity profile is symmetric. (This is characteristic of fully-developed, steady flow.) The shear stress is a linear profile and is related to the velocity profile in the following way:

$$\tau_{rz} = \mu \frac{\partial u}{\partial r} = \frac{r}{2}\frac{dp}{dz} \quad (3.5)$$

Notice the magnitude of the shear stress is greatest at the cylinder wall, where the velocity gradient is steepest, whereas the shear stress is zero at r = 0, where the velocity gradient is identically zero. Intuitively, this makes sense because the local velocity field around r = 0 is very similar in magnitude to the velocity at r = 0. There is a little relative motion between fluid "particles", thus little shear stress. At the wall, there is a steep gradient in the velocity profile. Small changes in radius are proportional to relatively large changes in velocity. Thus, we expect the highest shear stress to occur at the wall.

The total friction exerted by the wall on the fluid is the shear stress x the lateral surface area of the cylinder or:

$$F_{fric} = \tau_{rz}\rceil_{r=R} \cdot A = \frac{R}{2}\frac{dp}{dz} \cdot 2\pi RL \quad (3.6)$$

Note again this equation is only valid for fully-developed, steady, uni-directional flow, so that the L is based on cylinder length in this regime.

4.

a)

Irrotational flow states that the curl of the vector field is zero, or:

$$\nabla \times \vec{u} = 0 \quad (4.1)$$

From vector calculus, we know that if the curl of a vector is zero, then that vector can be written as the gradient of a scalar called the potential. The velocity potential Φ is thus:

$$\vec{u} = \nabla\Phi \quad (4.2)$$

Using the conservation of mass equation, we have:

$$\nabla \cdot \vec{u} = 0 = \nabla \cdot \nabla\Phi = \nabla^2\Phi = 0 \quad (4.3)$$

b)
1-dimensional potential flow written in Cartesian coordinates can be written as:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} = 0 \quad (4.4)$$

Integrating twice, we have:

$$\Phi(x) = Ax + B \quad (4.5)$$

Using the Boundary conditions, $\Phi(0) = 1$ and $\Phi(2) = 10$, the particular solution is thus:

$$\Phi(x) = \frac{9}{2}x + 1 \quad (4.6)$$

The velocity field is then:

$$\vec{u} = \frac{\partial}{\partial x}\big(\Phi(x)\big) = \frac{9}{2} \quad (4.7)$$

c)
We can conclude that 1-dimensional potential flow is uni-directional and uniform (not a function of space or time).