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5/4/11

MAE 4230/5230
SOLUTIONS TO HOMEWORK 10

<2> Stream function ψ

The velocity $\underline{u}(\underline{x},t)$ of an incompressible flow is divergence free ie., $\underline{\nabla} \cdot \underline{u}(\underline{x},t) = 0$.
Therefore, we can express $\underline{u}(\underline{x},t)$ in terms of a vector potential $\psi(\underline{x},t)\hat{z}$ by writing

$$\underline{u}(\underline{x},t) = \underline{\nabla} \times \psi(\underline{x},t)\hat{z}$$

(a)
$$\underline{\nabla} \times \psi(\underline{x},t)\hat{z} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial \psi}{\partial y}\right) + \hat{j}\left(-\frac{\partial \psi}{\partial x}\right) + \hat{k}(0)$$

$\therefore$ In 2D, $\boxed{u_x = \frac{\partial \psi}{\partial y} \quad u_y = -\frac{\partial \psi}{\partial x}}$

(b) Consider the equation of a stream ~~funct~~ line in 2D

$$\frac{dx}{u_x} = \frac{dy}{u_y}$$

$$\Rightarrow \quad \frac{dx}{\frac{\partial \psi}{\partial y}} = \frac{dy}{-\frac{\partial \psi}{\partial x}}$$

$$\Rightarrow \quad \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$\Rightarrow \quad d\psi = 0$$

$$\Rightarrow \quad \boxed{\psi = \text{constant}}$$

Hence, the equation of a streamline reduces to $\psi = \text{constant}$. Therefore, ψ is constant along a streamline and takes different values for different streamlines.

⟨3⟩ Complex potential

$$W(z) = \phi(x, y) + i\psi(x, y), \quad z = x + iy$$

$$U_x - iU_y = \frac{dW}{dz}$$

For potential flow past a cylinder of radius a, $W(z) = U(z + a^2/z)$

(a) $W(z) = U\left\{(x+iy) + \dfrac{a^2}{x+iy}\right\}$

$$= U\left[(x+iy) + \frac{a^2(x-iy)}{x^2+y^2}\right]$$

$\Rightarrow \quad W(z) = U\left[\left(x + \frac{a^2 x}{x^2+y^2}\right) + i\left(y - \frac{a^2 y}{x^2+y^2}\right)\right]$

$\Rightarrow \quad W(z) = \phi(x,y) + i\psi(x,y)$, where

$$\boxed{\begin{aligned} \phi(x,y) &= Ux\left[1 + \frac{a^2}{x^2+y^2}\right] \\ \psi(x,y) &= Uy\left[1 - \frac{a^2}{x^2+y^2}\right] \end{aligned}}$$

(b) Let us convert to polar coordinates for simpler calculation. We can also use cartesian without too much difficulty.

$$\psi(r,\theta) = Ur\sin\theta\left[1 - \frac{a^2}{r^2}\right]$$

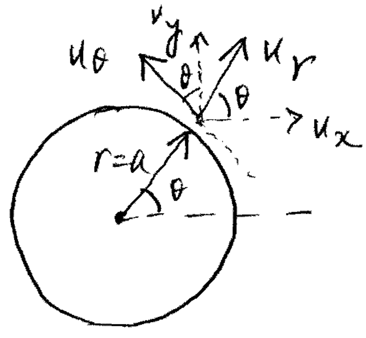
$\left\{ \begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2+y^2} \end{aligned} \right\}$

To show that $r=a$ is a streamline, it is suffice to show that $\psi(r=a) = \text{constant}$ based on problem <2>

$$\psi(r=a) = Ua\sin\theta\left[1 - \frac{a^2}{a^2}\right] = 0$$
$$= \text{constant}$$

$\Rightarrow \quad \underline{r=a \text{ is a streamline}}$

(c) $r=a$ is a streamline implies the normal velocity at the cylinder surface must be zero.



$$\underline{u}(r,\theta) = \underline{\nabla} \times \psi(r,\theta)\hat{k} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \frac{1}{r}\hat{e}_r\left(\frac{\partial \psi}{\partial \theta}\right) + \hat{e}_\theta\left(-\frac{\partial \psi}{\partial r}\right) + \hat{e}_z(0)$$

$$= \left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\hat{e}_r + \left(-\frac{\partial \psi}{\partial r}\right)\hat{e}_\theta$$

$$\therefore \; u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\left[Ur\sin\theta\left(1 - a^2/r^2\right)\right]$$

$$= \frac{1}{\cancel{r}} U\cancel{r}\cos\theta\left(1 - a^2/r^2\right)$$

$$\therefore \; \underline{u_r(r=a) = U\cos\theta\left(1 - a^2/a^2\right) = 0}$$

If we want to use cartesian coordinates

$$u_r = u_x\cos\theta + u_y\sin\theta$$

(d) Plot of the streamlines and a simple code to generate them in MATLAB is attached. We just need to make a x-y contour plot of the streamlines with each contour having a specific value of ψ i.e. representing a specific streamline. See next page.

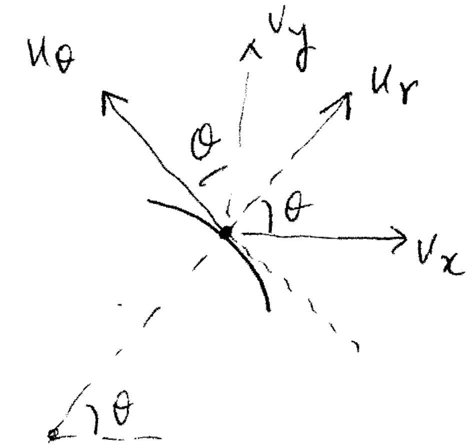
(e) Slip velocity $u_\theta = -\dfrac{\partial \psi}{\partial r}$

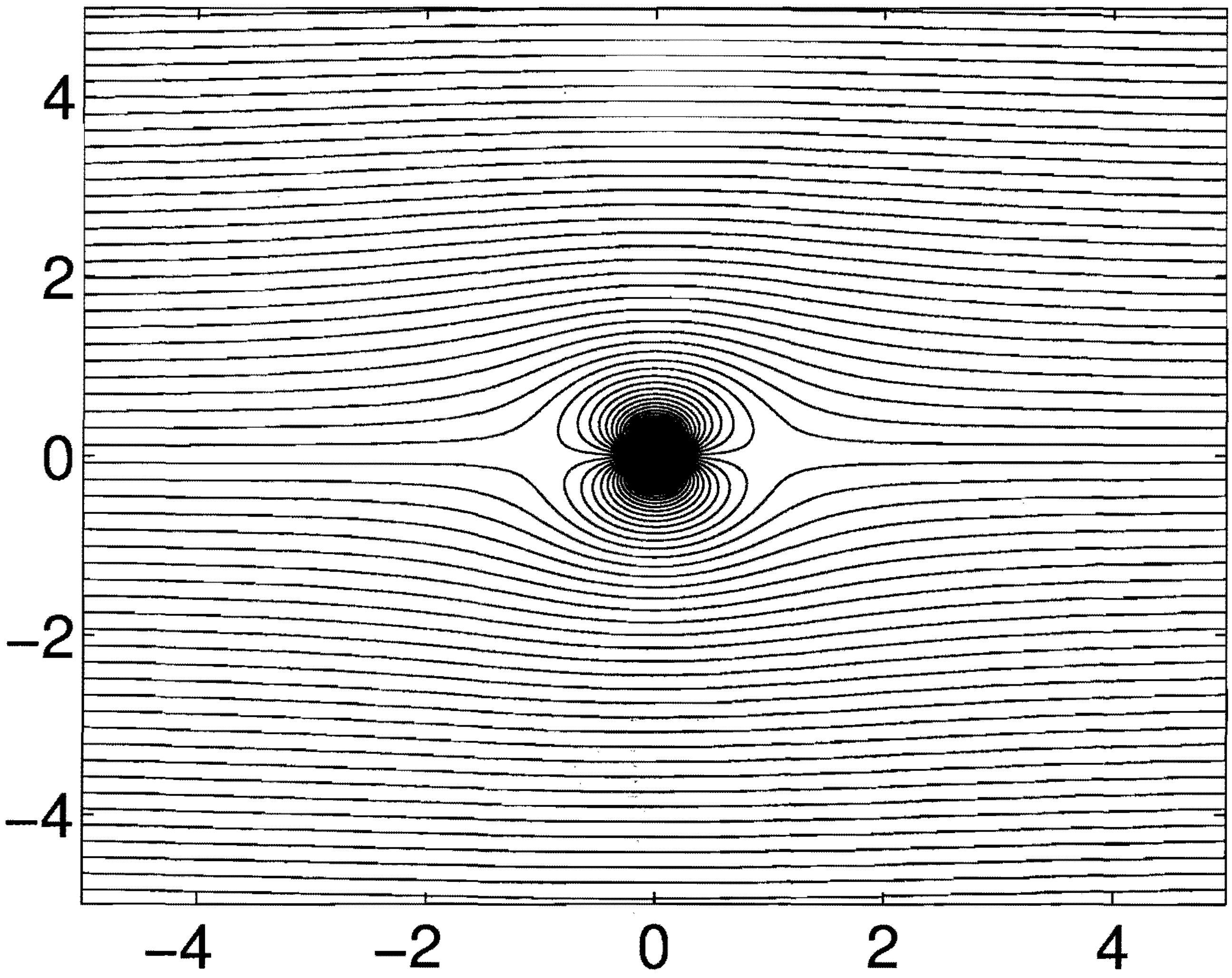
$$= -\frac{\partial}{\partial r}\left[U\sin\theta\left(r - a^2/r\right)\right]$$

$$= -U\sin\theta\left(1 + a^2/r^2\right)$$

$\therefore$ $\underline{u_\theta(r=a) = -2U\sin\theta} \neq 0$ $\underline{\text{for all } \theta}$

Again, if we want to use cartesian coordinates, $u_\theta = -u_x \sin\theta + v_y \cos\theta$





⟨3⟩(d)

Code in the next page

```
% PLOT THE STREAMLINES
% CONTOUR PLOT OF X-Y FOR DIFFERENT VALUES OF PSI

set(0,'defaulttextinterpreter','latex');
set(0,'defaulttextfontsize',24);
set(0,'defaultaxesfontsize',24);
set(0,'defaultLineLinewidth',1.5);

a=1;u=1;                                  % SET PARAMETERS RADIUS a AND
                                          %  FREE-STREAM VELOCITY U
[x,y]=meshgrid(-5:0.03:5,-5:0.03:5);      % SET UP GRID FOR CONTOUR PLOT
psi=u.*(y-(a^2).*y./(x.^2+y.^2));         % EXPRESSION FOR STREAM-FUNCTION
contour(x,y,psi,500)                      % CONTOUR PLOT WITH 500 CONTOUR LEVELS
                                          % FOR SMOOTHNESS
```

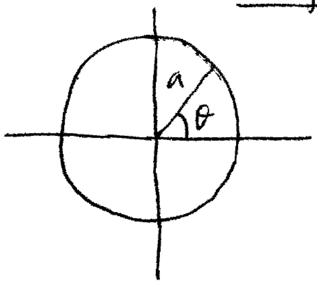
⟨4⟩ Conformal mapping

$z = x + iy$.

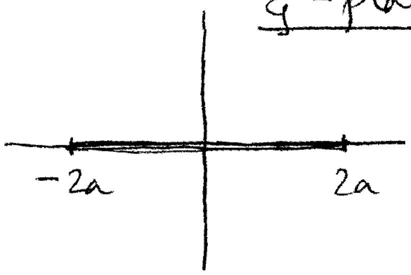
Map z to ξ: $\xi = \left(z + a^2/z\right)$

(a) A circle of radius $r = a$ in the z-plane $\Rightarrow$ $x = a\cos\theta$, $y = a\sin\theta$.

z-plane



ξ-plane



$$\therefore\ \xi = (x+iy) + \frac{a^2}{x+iy}$$

$$= x + iy + \frac{a^2(x-iy)}{x^2+y^2}$$

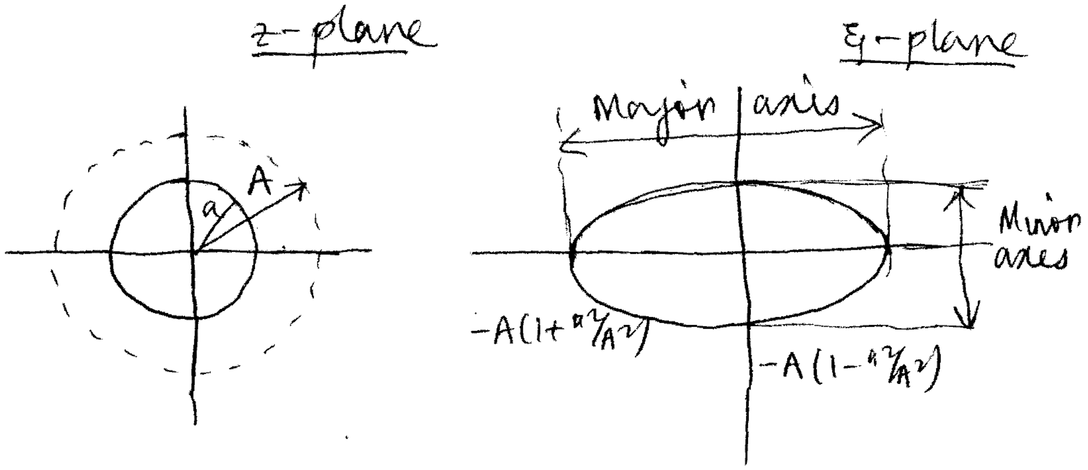
$$= x + iy + \frac{\cancel{a^2}(x-iy)}{\cancel{a^2}}$$

$$= 2x$$

A circle $r = a$ in z-plane $\Rightarrow$ x takes values $-a$ to a along the circle ie. $\underline{-a < x < a}$

$\Rightarrow$ $\underline{-2a < \xi < 2a}$ $\Rightarrow$ a line in the ξ-plane between $-2a$ and $2a$.

(b)



Consider now a circle of radius $A > a$ in the z-plane. Then, $x = A\cos\theta$, $y = A\sin\theta$

$$\therefore\ \xi = (x+iy) + \frac{a^2(x-iy)}{A^2}$$

$$= x\left(1+\frac{a^2}{A^2}\right) + iy\left(1-a^2/A^2\right)$$

$$= X + iY$$

where $X = x(1+a^2/A^2) = A\cos\theta\,(1+a^2/A^2)$
$Y = y(1-a^2/A^2) = A\sin\theta\,(1-a^2/A^2)$

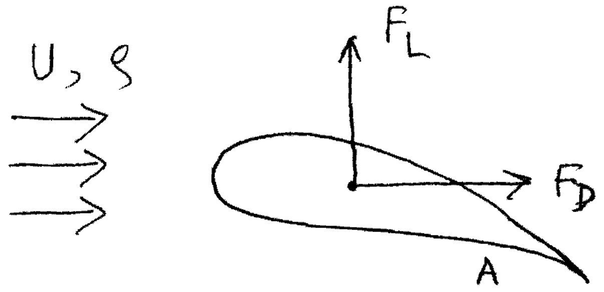
$$\Rightarrow \left[\frac{X}{A(1+a^2/A^2)}\right]^2 + \left[\frac{Y}{A\left(1-\frac{a^2}{A^2}\right)}\right]^2 = \cos^2\theta + \sin^2\theta$$

$$= 1$$

$\Rightarrow$ an ellipse in $\xi(X-Y)$ plane with major and minor axes $2A(1+a^2/A^2)$ and $2A(1-a^2/A^2)$ respectively.

⟨5⟩ Lift and Drag Coefficient

(a)



We can express both the lift and the drag forces as

$$F_L = C_L\left(\tfrac{1}{2}\rho U^2 A\right)$$
$$F_D = C_D\left(\tfrac{1}{2}\rho U^2 A\right)$$

where C_L, C_D would be potentially functions of the airfoil parameters. Therefore,

$$\boxed{F \propto \rho U^2 A}$$

Dimension: $\rho U^2 A = \dfrac{M}{L^3}\cdot\left(\dfrac{L}{T}\right)^2 L^2$

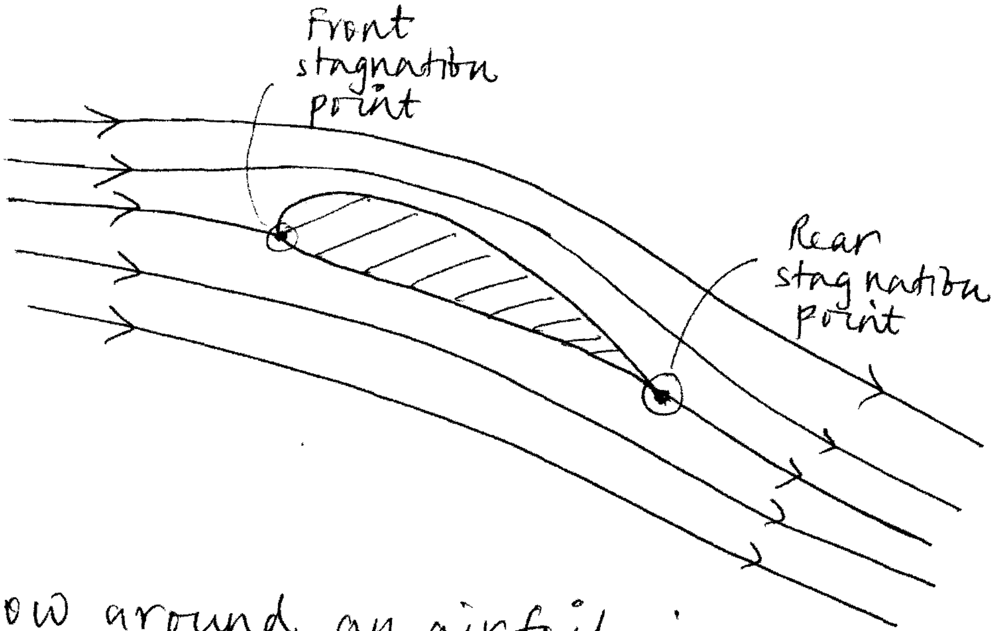
$= \underline{MLT^{-2}} = \boxed{\text{force}}$

In 2D, considering unit width our forces would scale as $\boxed{F \propto \rho U^2 L(1)}$

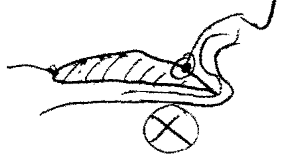
Dimension: $\rho U L(1) = \dfrac{M}{L^3}\left(\dfrac{L}{T}\right)^2 L\cdot(L)$

$= \underline{MLT^{-2} = \text{force}}$

(b)

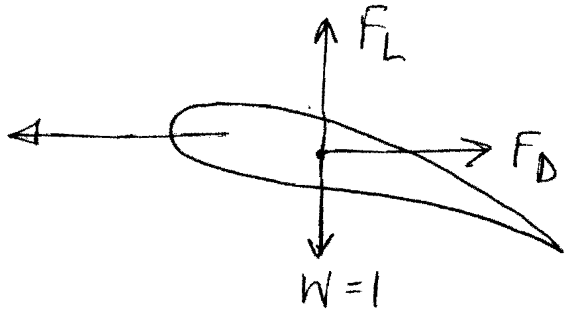


The flow around an airfoil is sketched (ideal flow) in the figure above. The most important feature of this flow field is the location of the rear stagnation point at the trailing edge of the airfoil. This is fixed by the Kutta condition, which thereby attempts to incorporate some effect of fluid viscosity in this ideal flow scenario. By fixing the location of the rear stagnation point at the trailing edge, it avoids the unphysical situation of the fluid from the bottom of the airfoil making a sharp turn around the trailing edge, which is



allowed by ideal flow, but not by viscosity which will smoothen any such sharp gradients in the flow.

(c)



Efficiency η = (work required to transport a unit weight $W=1$ by a unit distance $d=1$)$^{-1}$

$$= \left(\frac{F_D}{W} \cdot d \right)^{-1} = \left(\frac{W}{F_D} \right)$$

In <u>steady translation</u>, $W = F_L$.

$$\Rightarrow \boxed{\eta = \frac{F_L}{F_D} = \frac{\text{Lift}}{\text{Drag}}}$$