Let us assume that, as is usual, dp/dx is given as a function of x, and, further, that the velocity u is given as a function of y at some initial value of x. Then we can determine numerically, from each u, the associated $\partial u/\partial x$, and with one of the known algorithms we can then proceed, step by step, in the x-direction. A difficulty exists, however, in various singularities which appear on the fixed boundary. The simplest case of the flow situations under discussion is that of water streaming along a thin flat plate. Here a reduction in variables is possible; we can write $u = f(y/x^{\frac{1}{2}})$. By numerical integration of the resulting differential equation we obtain an expression for the drag

$$D = 1.1 \times b\sqrt{\mu\rho l u_0^3}$$

(b breadth, l length of the plate, u_0 velocity of the undisturbed water relative to the plate). The velocity profile is shown in [Fig. 8.1].

For practical purposes the most important result of these investigations is that in certain cases, and at a point wholly determined by the external conditions, *the flow separates from the wall* [Fig. 8.2]. A fluid layer which is set into rotation by friction at the wall thus pushes itself out into the free fluid where, in causing a complete transformation of the motion, it plays the same role as a Helmholtz surface of discontinuity. A change in the coefficient of viscosity μ produces a change in the thickness of the vortex layer (this thickness being proportional to $\sqrt{(\mu l/\rho u)}$), but everything else remains unchanged, so that one may, if one so wishes, take the limit $\mu \to 0$ and still obtain the same flow picture.

Separation can only occur if there is an increase in pressure along the wall in the direction of the stream . . .

The amount of insight packed into this part of Prandtl's paper is staggering, and much of the present chapter will be spent filling in the details, particularly with regard to the derivation of the

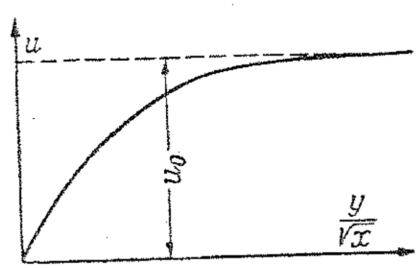

Fig. 8.1. Prandtl's diagram of the velocity profile in the boundary layer on a flat plate.

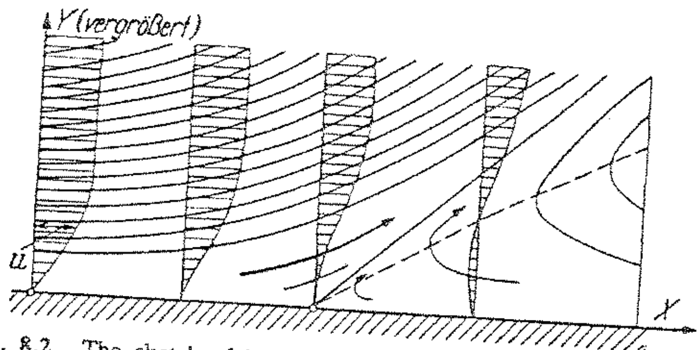

Fig. 8.2. The sketch of boundary-layer separation in Prandtl's 1905 paper.

boundary layer equations (§8.2) and their solution in the case of flow past a flat plate (§8.3).

After the passage quoted above, Prandtl emphasizes how the flow of a fluid of small viscosity must be dealt with in two interacting parts, namely an inviscid flow obeying Helmholtz's vortex theorems and thin boundary layers in which viscous effects are important. The motion in the boundary layers is regulated by the pressure gradient in the mainstream flow but, on the other hand, the character of the mainstream flow is, in turn, markedly influenced by any separation that may occur.

Prandtl goes on to discuss some particular cases, including the impulsively started motion of a circular cylinder (Fig. 8.3). He finally reports some experiments undertaken in a hand-operated

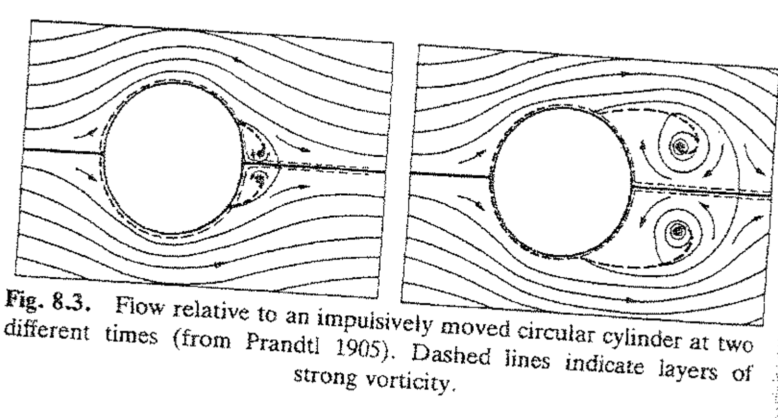

Fig. 8.3. Flow relative to an impulsively moved circular cylinder at two different times (from Prandtl 1905). Dashed lines indicate layers of strong vorticity.

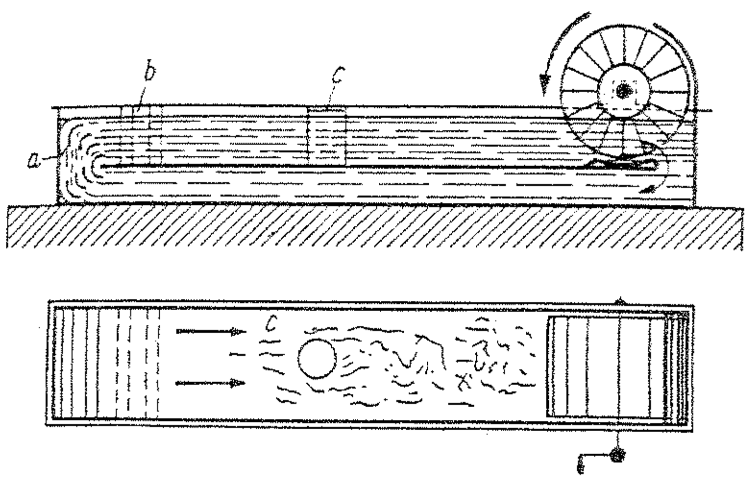

Fig. 8.4. Prandtl's hand-operated flow tank.

water tank (Fig. 8.4). These include flow past a wall, flow past a circular arc at zero incidence, and flow past a circular cylinder. In the last case he demonstrates that even a very small amount of suction into a slit on one side of the cylinder is enough to prevent separation of the boundary layer on that side (Fig. 8.5). He notes, too, a most interesting consequence of this, because 'the speed must decrease in the broadening aperture through which the water flows, and therefore the pressure must rise'. A substantial adverse pressure gradient will therefore be impressed on the boundary layer on the corresponding side wall of the tank

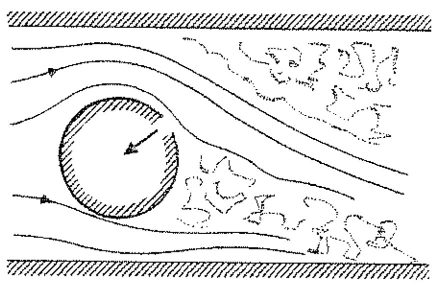

Fig. 8.5. Sketch of the final photograph in Prandtl's paper.

(uppermost in Fig. 8.5), and accordingly *that* boundary layer must be expected to separate. Such separation is indeed observed (Fig. 8.5), and on this successful note the paper ends.†

For all its fundamental insights, the paper was scarcely an overnight success, and several years were to pass before Prandtl's work became widely known outside Germany, let alone fully appreciated.

Prior to 1900, ideal flow theory and viscous flow theory had more or less gone their separate ways. On the inviscid side there had been the great papers of Euler (1755) on the fundamental equations, of Helmholtz (1858) on vortex motion and of Kelvin (1869) on the circulation theorem. There had been success, too, in accounting for many of the most important properties of water waves and sound waves. There was no doubt, then, that inviscid flow theory had its value. On the other hand, it was well known that uniform flow past a 'bluff' body—such as a circular cylinder—bore little resemblance *at the rear of the body* to the predictions of inviscid flow theory.

Viscous flow theory effectively began with the pioneering paper of Stokes (1845), who not only laid down the equations of motion but obtained many of the elementary exact solutions that are to be found in Chapter 2. He followed this with another important paper (1851) on what we would now call low Reynolds number flow (§7.2), and when Hele-Shaw performed his remarkable experiments with irrotational streamline patterns (1898; see §7.7) it was Stokes again who produced the relevant viscous theory. The other pioneering work of the time was by Reynolds, notably his beautiful experiments in 1883 on transition in flow down a pipe (§9.1).

Yet a major problem remained: that of accounting for the motion of a fluid of small viscosity past a solid body. Prandtl was not alone, of course, in addressing the matter. As early as 1872 Froude had conducted experiments on the drag on a thin flat plate towed through still water, and had attributed that drag to

† Prandtl's paper is not exclusively concerned with boundary layers; its subject is the motion of a fluid with very small viscosity. He points out, quite early in the paper, a wholly different way in which small viscosity can be significant, namely through its cumulative effect, over a long time interval, in a region of closed streamlines (see §5.10).

the layers of fluid in intense shear near the plate. He had found, too, that the drag varied not in proportion to the length l of the plate, but at a slower rate. Lanchester later proposed, independently of Prandtl, that the drag should be proportional to $\mu^{1/2}u_0^{3/2}$. He also discussed separation, and affirmed correctly that on a rotating cylinder in a uniform stream separation would be delayed on one side and hastened on the other. He published all this and much more, in his *Aerodynamics* of 1907, although just how much earlier the work was done is not entirely clear.

If, nigh on a hundred years later, the concept of a boundary layer and its separation seem to have been a long time in the making, it is worth recalling that there were at least two factors which clouded the issue at the time. First, there was substantial uncertainty about whether the correct boundary condition was one of no slip. It is one thing to find Stokes unsure about the matter on pp. 96–99 of his 1845 paper, but it is quite another to find this uncertainty continuing right up to the turn of the century, with some investigators convinced of the no-slip condition only in the case of slow flow (see Goldstein 1969, and pp. 676–680 of Goldstein 1938). Second, it was known that when ideal flow theory predicts a negative value of the absolute pressure at any point in a liquid, the formation of bubbles of vapour, known as *cavitation*, may be expected. Thus when irrotational flow past sharp corners (e.g. Fig. 4.6(a)) bore little resemblance to the actual flow of real liquids such as water, there seemed to be a ready explanation: ideal flow theory implies an infinite speed at the corner, and by Bernoulli's theorem this means an infinitely negative pressure. The onset of cavitation will prevent such a singularity occurring but, in so doing, will give rise to a different and 'separated' flow (see, e.g., Batchelor 1967, pp. 497–506). In an essay on pp. 1–5 of Rosenhead (1963), Lighthill emphasizes how this obscured the possibility that there might be a quite different (viscous) explanation for flow separation, one which would obtain, indeed, for liquids or gases and whether the rigid boundary was sharp-cornered or smooth. It was this quite different explanation, along with so much else, that Prandtl was eventually to squeeze into just a few pages in 1904.

8.2. The steady 2-D boundary layer equations

We now derive the equations for a steady 2-D boundary layer adjacent to a rigid wall $y = 0$:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{8.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{8.2}$$

$p(x)$ being a function of x alone. The boundary conditions at the wall are

$$u = v = 0 \qquad \text{at } y = 0, \tag{8.3}$$

if the wall is at rest. The boundary layer flow must also match with the inviscid mainstream in some appropriate manner. This is a matter of some subtlety, and we postpone it for the time being.

There are two key ideas involved in boundary layer theory. The first is that u and v vary much more rapidly with y, the coordinate normal to the boundary, than they do with x, the coordinate parallel to the boundary. Let U_0 denote some typical value of u, and let u change by an amount of order U_0 over an x-distance of order L, say. If δ denotes a typical value of the thickness of the boundary layer, our basic approximation is

$$\left|\frac{\partial u}{\partial y}\right| \gg \left|\frac{\partial u}{\partial x}\right|,$$

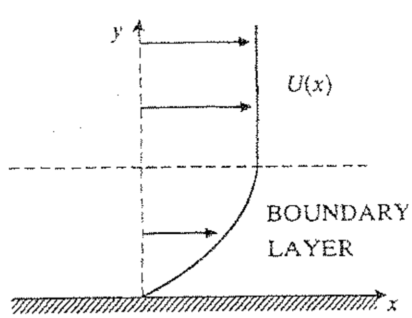

Fig. 8.6. The boundary layer.

and this amounts, by making an order of magnitude estimate of each term, to $U_0/\delta \gg U_0/L$, i.e.

$$\delta \ll L. \tag{8.4}$$

Now, the exact 2-D equations are

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{8.5}$$

It follows at once from the last of these that $\partial v/\partial y$ is of order U_0/L, and as v is zero at $y = 0$ it follows that v is of order $U_0\delta/L$ in the boundary layer. Thus v is much smaller than u. On viewing the first two equations as expressions for $\partial p/\partial x$ and $\partial p/\partial y$ respectively it then follows that

$$\left|\frac{\partial p}{\partial y}\right| \ll \left|\frac{\partial p}{\partial x}\right|,$$

which means that in the boundary layer p is, to a first approximation, a function of x alone. This justifies the use of $\mathrm{d}p/\mathrm{d}x$, rather than $\partial p/\partial x$, in eqn (8.1), and bears out Prandtl's remark that 'the pressure distribution of the free fluid will be impressed on the transition layer'. But the most dramatic simplification of eqn (8.5) arises on account of the following estimates:

$$\frac{\partial^2 u}{\partial x^2} = O\left(\frac{U_0}{L^2}\right), \qquad \frac{\partial^2 u}{\partial y^2} = O\left(\frac{U_0}{\delta^2}\right). \tag{8.6}$$

In view of eqn (8.4) the term $\partial^2 u/\partial x^2$ is negligible compared with the term $\partial^2 u/\partial y^2$, and with this major simplification of eqn (8.5) we complete our derivation of eqn (8.1).

The other key idea involved in boundary layer theory is that the rapid variation of u with y should be just sufficient to prevent the viscous term from being negligible, notwithstanding the small coefficient of viscosity ν. We may at once use this consideration to obtain an order of magnitude estimate of the boundary layer thickness. Both non-linear terms on the left-hand side of eqn (8.1) are of order U_0^2/L, and in order that the viscous term be of comparable magnitude we require that

$$\frac{U_0^2}{L} \sim \frac{\nu U_0}{\delta^2},$$

i.e.

$$\frac{\delta}{L} = O(R^{-\frac{1}{2}}). \tag{8.7}$$

The basic hypothesis (8.4) is evidently correct if the Reynolds number $R = U_0 L/\nu$ is large; the whole procedure is then self-consistent, and may indeed be put on a more formal basis (Exercise 8.1).

Equation (8.1) is also valid in the case of a curved boundary, provided that x denotes distance along the boundary and y distance normal to it. This may be demonstrated by writing the full Navier–Stokes equations in a suitable system of curvilinear coordinates (x, y); the argument is much as before, save that $\partial p/\partial y$ is then comparable in magnitude with $\partial p/\partial x$, for a substantial pressure gradient in the y-direction is required to balance the centrifugal effect of the flow round the curved surface (Rosenhead 1963, pp. 201–203; Goldstein 1938, pp. 119–120). It is still the case that within the boundary layer p is essentially a function of x alone, for although the two pressure gradients are comparable, actual changes in p across the boundary layer are still much smaller, by a factor $O(\delta/L)$, than changes in p along the boundary, simply because the boundary layer is so thin.

To actually determine the pressure distribution $p(x)$, suppose that $U(x)$ denotes the slip velocity that would arise, at $y = 0$, if the fluid were (mistakenly) treated as being entirely inviscid. The velocity at the 'edge' of the boundary layer in Fig. 8.6 will be almost equal to $U(x)$, and by Bernoulli's theorem $p + \frac{1}{2}\rho U^2$ will be constant along a streamline at the edge of the boundary layer. It follows that

$$-\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = U\frac{\mathrm{d}U}{\mathrm{d}x}; \tag{8.8}$$

thus if $U(x)$ increases with x the pressure $p(x)$ decreases, and vice versa.

We must finally address the matter of how to ensure a 'match' between the flow velocity in the boundary layer and that in the inviscid mainstream. In the sections which follow we shall, to this end, impose on the boundary layer flow the condition

$$u \to U(x) \qquad \text{as } y/\delta \to \infty, \tag{8.9}$$

δ denoting a typical measure of the boundary layer thickness, proportional to $\nu^{\frac{1}{2}}$. Note that the limiting process here is $y/\nu^{\frac{1}{2}} \to \infty$, not $y \to \infty$, which would correspond to rocketing out of the laboratory and into the heavens. This important distinction may become clearer with the following elementary example.

An elementary differential equation with a 'boundary layer'

Consider the following problem for a function $u(y)$:

$$\varepsilon u'' + u' = 1; \qquad u(0) = 0, \quad u(1) = 2, \tag{8.10}$$

where ε denotes a small positive constant. The exact solution is easily shown to be

$$u = y + \frac{1 - e^{-y/\varepsilon}}{1 - e^{-1/\varepsilon}}. \tag{8.11}$$

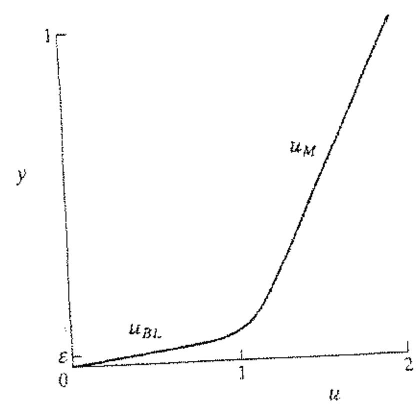

Fig. 8.7. The solution to eqn (8.10) for small ε.

Now, $e^{-1/\varepsilon}$ is extremely small, and so is $e^{-y/\varepsilon}$, for $0<y<1$, unless y is of order ε. The solution may therefore be approximated, in two parts, by a 'mainstream'

$$u_M = y + 1,$$

and a 'boundary layer' adjacent to $y = 0$ with thickness of order ε:

$$u_{BL} = 1 - e^{-y/\varepsilon}.$$

These two expressions represent particular limits of the full, exact solution (8.11), the first being obtained by letting $\varepsilon \to 0$ at fixed y, and the second being obtained by letting $\varepsilon \to 0$ *with* y/ε *fixed*. Notably,

$$\lim_{y/\varepsilon \to \infty} u_{BL} = \lim_{y \to 0} u_M,$$

and this is the equivalent statement to eqn (8.9) in this elementary example.

It is instructive to take the analogy further by returning to eqn (8.10) and proceeding on an approximate basis from the outset, exploiting the fact that ε is small. If we neglect the term $\varepsilon u''$ entirely, on this basis, we obtain

$$u_0' = 1, \qquad \text{i.e. } u_0 = y + c,$$

and on making this satisfy the condition $u_0(1) = 2$ we obtain an 'outer' solution,

$$u_0(y) = y + 1.$$

This procedure thus far is comparable with treating a high Reynolds number flow as being entirely inviscid; the small parameter ε multiplies the highest derivative in the equation, and by ignoring that term we lower the order of the system and are unable to satisfy all the boundary conditions. Here an 'inner' solution, or boundary layer, is needed near $y = 0$, in order to satisfy the boundary condition there. We may recognize variations of u in this boundary layer to be much more rapid than those elsewhere by changing the independent variable in eqn (8.10) to

$$Y = y/\varepsilon.$$

With this scaling the previously negligible second derivative regains its importance:

$$\varepsilon \frac{1}{\varepsilon^2}\frac{d^2u}{dY^2} + \frac{1}{\varepsilon}\frac{du}{dY} = 1,$$

so that to a first approximation the inner solution u_i satisfies

$$\frac{d^2u_i}{dY^2} + \frac{du_i}{dY} = 0.$$

This is the equivalent of the boundary layer equation (8.1), in our simple example (and cf. Exercise 8.1). On making the inner solution satisfy the boundary condition $u(0) = 0$ we obtain

$$u_i = A(1 - e^{-Y}),$$

and the matching condition

$$\lim_{Y\to\infty} u_i = \lim_{y\to 0} u_0$$

determines that $A = 1$. Thus

$$u = \begin{cases} y + 1 & \text{as } \varepsilon \to 0 \text{ for fixed } y, \\ 1 - e^{-y/\varepsilon} & \text{as } \varepsilon \to 0 \text{ for fixed } y/\varepsilon, \end{cases}$$

in keeping with our deductions from the exact solution (8.11).

8.3. The boundary layer on a flat plate

On inviscid theory a uniform stream approaching a flat plate at zero angle of incidence is unaffected by the presence of the plate, so $U(x)$ is a constant. The boundary layer equations then reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{8.12}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{8.13}$$

We seek a similarity solution in which u is some function of the single variable

$$\eta = y/g(x). \tag{8.14}$$

This implies that the velocity profile at any distance x from the leading edge will be just a 'stretched out' version of the velocity profile at any other distance x, as in Fig. 2.14; this is a natural assumption if, as we shall suppose, the plate is semi-infinite, from $x=0$ to $x=\infty$. We here take the similarity method of §2.3 a little further by not attempting to guess the function $g(x)$ in advance; we show instead how it can be left to emerge in a rational way as the calculation proceeds.

We first satisfy eqn (8.13) by introducing a stream function $\psi(x, y)$ such that

$$u = \partial\psi/\partial y, \qquad v = -\partial\psi/\partial x. \tag{8.15}$$

If we write u in the form $u = Uh(\eta)$ we may integrate to obtain

$$\psi = Ug(x)\int_0^\eta h(s)\,\mathrm{d}s + k(x).$$

But we want the plate itself to be a streamline, so that $\psi = 0$, say, at $\eta = 0$; so $k(x) = 0$. It is then more convenient to write ψ in the form

$$\psi = Ug(x)f(\eta), \qquad \text{with } f(0) = 0, \tag{8.16}$$

whence

$$u = Uf'(\eta) \tag{8.17}$$

and

$$\begin{aligned} v = -\frac{\partial\psi}{\partial x} &= -U\left(g'f + gf'\frac{\partial\eta}{\partial x}\right) \\ &= -U\left(g'f - \frac{y}{g}f'g'\right) \\ &= U(\eta f' - f)g'. \end{aligned} \tag{8.18}$$

Here, of course, f' denotes $f'(\eta)$, but g' denotes $g'(x)$. On substituting for u and v in eqn (8.12) we obtain

$$-U^2 f'f''\frac{y}{g^2}g' + U^2(\eta f' - f)g'\frac{f''}{g} = \nu U\frac{f'''}{g^2},$$

which simplifies to

$$f''' + \frac{Ugg'}{\nu}ff'' = 0.$$

Our aim is, of course, to obtain an ordinary differential equation for f as a function of η. We must therefore choose gg'—which would otherwise be a function of x—to be a constant. Clearly the choice of ν/U for this constant is convenient in that it rids the equation of all parameters of the problem, and integrating $gg' = \nu/U$ gives

$$\tfrac{1}{2}g^2 = \frac{\nu x}{U} + d,$$

where d is an arbitrary constant. Now, if g vanishes for some value of x, certain flow quantities such as

$$\partial u/\partial y = Uf''/g$$

become singular. We clearly expect some such behaviour at the leading edge, if only because on $y=0$ the velocity suddenly changes from U in $x<0$ to zero in $x>0$. We therefore choose $d=0$ to ensure that any such behaviour occurs at the leading edge. Thus $g(x) = (2\nu x/U)^{\frac{1}{2}}$ and, to sum up, we have found that

$$\psi = (2\nu Ux)^{\frac{1}{2}} f(\eta), \qquad \text{where } \eta = \frac{y}{(2\nu x/U)^{\frac{1}{2}}}, \tag{8.19}$$

and

$$f''' + ff'' = 0. \tag{8.20}$$

This equation must be supplemented by the boundary conditions

$$f(0) = f'(0) = 0, \qquad f'(\infty) = 1. \tag{8.21}$$

The first of these stems from eqn (8.18), the second from eqn (8.17), and the third from the fact that u must tend to U, the mainstream value, as we leave the boundary layer (cf. eqn (8.9)).

The boundary value problem (8.20), (8.21) has to be solved numerically, and the results are shown in Fig. 8.8. The ratio u/U is 0.97 at $\eta=3$ and 0.999936 at $\eta=5$. According to eqn (8.19), therefore, the boundary layer thickness δ is such that

$$\delta = O\left(\frac{\nu x}{U}\right)^{\frac{1}{2}}, \tag{8.22}$$

as indicated in Fig. 2.14. As the boundary layer thickens the horizontal stress on the plate

$$t_x = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{y=0} = \mu \frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U\left(\frac{U}{2\nu x}\right)^{\frac{1}{2}} f''(0) \tag{8.23}$$

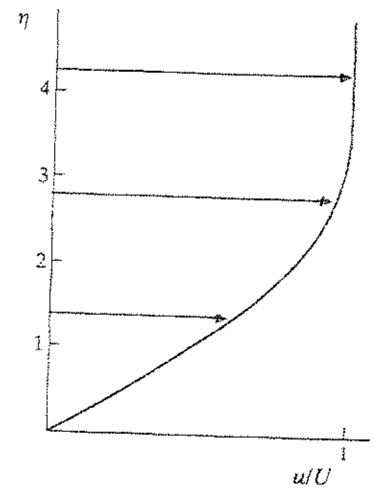

Fig. 8.8. The velocity profile in the boundary layer on a flat plate.

decreases with x. (Here we have used eqns (6.7) and (6.9), with, of course, $\boldsymbol{n} = (0, 1, 0)$.)

Application of the theory to a finite flat plate of length L

It is natural to hope that the above similarity solution will hold reasonably well for a finite plate of length L, even if behaviour of a different kind must be expected near and beyond the trailing edge. Taking into account both the top and the bottom of the plate, we obtain for the drag

$$D = 2\int_0^L t_x \, \mathrm{d}x = 2\sqrt{2}f''(0)\rho U^2 L R^{-\frac{1}{2}}, \qquad (8.24)$$

where $R = UL/\nu$. Thus D is proportional to $L^{\frac{1}{2}}$, rather than to L, because the velocity gradients at the plate decrease with x, corresponding to the thickening of the boundary layer. The drag is proportional to $\nu^{\frac{1}{2}}$, and vanishes as $\nu \to 0$. The numerical value of $f''(0)$ is 0.4696.

The agreement between boundary layer theory and experiment is very good, both in respect of the expression (8.24) for the drag and in respect of the details of the velocity profile. This agreement does break down, however, if the Reynolds number is