HW6 Solutions

2) Boundary Layer equations in non-dimensional forms

Rewrite the exact 2D equations of motion in terms of the non-dimensional and scaled variables

\[ x' = \frac{x}{L}, \quad y' = \frac{y}{\text{Re}^{1/2} L}, \quad u' = \frac{u}{U_0}, \quad v' = \frac{v}{\text{Re}^{1/2} U_0}, \quad p' = \frac{p}{\rho U_0^2}, \]

where \( \text{Re} = U_0 L / v \). By taking the limit \( \text{Re} \to \infty \) with fixed \( u' \), \( \partial u' / \partial x' \), etc., derive the boundary layer equations in their non-dimensional and scaled form in steady state:

\[
\begin{align*}
    u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} &= -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2}, \\
    0 &= -\frac{\partial p'}{\partial y'}, \quad \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0.
\end{align*}
\]

Solution:
This problem is an exercise in non-dimensionalization. Chain-rule will be used heavily. Our task is to rewrite the steady 2-D Navier-Stokes Equation and Continuity equation in a non-dimensionalized form, take the limit as Reynolds number goes to infinity, and thereby derive the above non-dimensionalized boundary layer equations.

Incompressible Continuity Equation:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

2-D steady Navier-Stokes Equation:

\[
\begin{align*}
    \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) &= -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\
    \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) &= -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right).
\end{align*}
\]
We have explicit expressions of \( u \) and \( v \) and in terms of non-dimensional velocities \( u' \) and \( v' \), but the derivative terms are a little more tricky. As a goal, we want to write each dimensional derivative in terms of a non-dimensional derivative. So for example, let’s write \( \frac{\partial u}{\partial x} \) in terms of \( \frac{\partial u'}{\partial x'} \). Using chain rule we have:

\[
\frac{\partial u}{\partial x} = \frac{\partial u}{\partial u'} \cdot \frac{\partial u'}{\partial x'} \cdot \frac{\partial x'}{\partial x} = \left( \frac{U_o}{L} \right) \frac{\partial u'}{\partial x'}
\]

The 2\textsuperscript{nd} derivatives are a little tricky:

\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{U_o \partial u'}{L \partial x'} \right) = \frac{U_o \partial u'}{L \partial x'} = \frac{U_o \partial^2 u'}{L \partial x'^2} \cdot \frac{1}{L} = \left( \frac{U_o}{L^2} \right) \frac{\partial^2 u'}{\partial x'^2}
\]

Here, \( \frac{\partial u'}{\partial x'} \) was replaced with \( u'_x \), read as the derivative of \( u' \) with respect to \( x' \). This may make the notation a little simpler to follow. Applying the same process to the other derivative terms in the 2-D steady Navier-Stokes Equation, we find:

\[
\frac{\partial u}{\partial y} = \left( \frac{U_o}{Re^{-\frac{1}{2}} L} \right) \frac{\partial u'}{\partial y'} \text{ and } \frac{\partial^2 u}{\partial y^2} = \left( \frac{U_o}{Re^{-1} L^2} \right) \frac{\partial^2 u'}{\partial y'^2}
\]

\[
\frac{\partial v}{\partial x} = \left( \frac{Re^{-\frac{1}{2}} U_o}{L} \right) \frac{\partial v'}{\partial x'} \text{ and } \frac{\partial^2 v}{\partial x^2} = \left( \frac{Re^{-\frac{1}{2}} U_o}{L^2} \right) \frac{\partial^2 v'}{\partial x'^2}
\]

\[
\frac{\partial v}{\partial y} = \left( \frac{U_o}{L} \right) \frac{\partial v'}{\partial y'} \text{ and } \frac{\partial^2 v}{\partial y^2} = \left( \frac{U_o}{Re^{-\frac{1}{2}} L^2} \right) \frac{\partial^2 v'}{\partial y'^2}
\]

\[
\frac{\partial p}{\partial x} = \left( \frac{\rho U_o^2}{L} \right) \frac{\partial p'}{\partial x'} \text{ and } \frac{\partial p}{\partial y} = \left( \frac{\rho U_o^2}{Re^{-\frac{1}{2}} L} \right) \frac{\partial p'}{\partial y'}
\]
Using these derivative expressions and the definitions of non-dimensional quantities given in the original problem statement, plug these into the continuity equation and 2-D steady Navier-Stokes Equations:

**Continuity:**

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

\[
\left[ \left( \frac{U_o}{L} \right) \frac{\partial u'}{\partial x'} \right] + \left[ \left( \frac{U_o}{L} \right) \frac{\partial v'}{\partial y'} \right] = 0
\]

\[
\therefore \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0
\]

**x-momentum:**

\[
\rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

\[
\rho \left( U_o u' \cdot \left[ \left( \frac{U_o}{L} \right) \frac{\partial u'}{\partial x'} \right] + Re^{-\frac{1}{2}} U_o v' \cdot \left[ \left( \frac{U_o}{Re^{-\frac{1}{2}} L} \right) \frac{\partial u'}{\partial y'} \right] \right) = -\left( \rho U_o^2 \frac{L^2}{Re} \right) \frac{\partial p'}{\partial x'} + \mu \left[ \left( \frac{U_o}{L} \right) \frac{\partial^2 u'}{\partial x'^2} \right] + \left( \frac{U_o}{Re^{-\frac{1}{2}} L} \right) \frac{\partial^2 u'}{\partial y'^2}
\]

\[
\frac{\rho U_o^2}{L^2} \left( u' \cdot \frac{\partial u'}{\partial x'} + v' \cdot \frac{\partial u'}{\partial y'} \right) = -\left( \frac{\rho U_o^2}{L} \right) \frac{\partial p'}{\partial x'} + \rho U_o \left[ \left( \frac{\mu}{\rho L U_o} \right) \frac{\partial^2 u'}{\partial x'^2} \right] + \left( \frac{1}{Re^{-\frac{1}{2}} \cdot Re} \right) \frac{\partial^2 u'}{\partial y'^2}
\]

\[
u' \cdot \frac{\partial u'}{\partial x'} + v' \cdot \frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{1}{Re} \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}
\]

Taking the limit as \( Re \to \infty \), we have:

\[
\therefore u' \cdot \frac{\partial u'}{\partial x'} + v' \cdot \frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2}
\]
y-momentum:

\[
\rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

\[
\rho \left( u_o u' \cdot \left( \frac{Re^{-\frac{1}{2}} U_o}{L} \frac{\partial v'}{\partial x'} \right) + Re^{-\frac{1}{2}} U_o v' \cdot \left( \frac{U_o}{L} \frac{\partial v'}{\partial y'} \right) \right) = - \left( \rho U_o^2 \frac{1}{Re^{-\frac{1}{2}} L} \frac{\partial p'}{\partial y'} \right) + \mu \left( \frac{Re^{-\frac{1}{2}} U_o}{L^2} \frac{\partial^2 v'}{\partial x'^2} \right) + \left( \frac{1}{Re^{1/2}} \frac{\partial^2 v'}{\partial y'^2} \right)
\]

\[
\frac{\rho U_o^2}{L} Re^{-\frac{1}{2}} \left( u' \cdot \frac{\partial v'}{\partial x'} + v' \cdot \frac{\partial v'}{\partial y'} \right) = - \left( \frac{\rho U_o^2}{Re^{-\frac{1}{2}} L} \frac{1}{L} \frac{\partial p'}{\partial y'} \right) + \mu \left( \frac{Re^{-\frac{1}{2}} U_o}{L} \frac{\partial^2 v'}{\partial x'^2} \right) + \left( \frac{1}{Re^{1/2}} \frac{\partial^2 v'}{\partial y'^2} \right)
\]

Taking the limit as \( Re \to \infty \), we have:

\[
\therefore \ 0 = -\frac{\partial p'}{\partial y'}
\]

3) Matlab solution of the similarity equation

For flow past a plate, the similarity solution \( f(\eta) \) is governed by

\[
f'''' + ff'' = 0, \text{ with boundary conditions } f(0) = f'(0) = 0, f'(\infty) = 1.
\]

Solve \( f(\eta) \) numerical using Matlab and plot your solution.

[Hint: you can define \( g(\eta) = f'(\eta), h(\eta) = g'(\eta) \) to reduce the 3rd order differential equation into three first order differential equations.]
Solution:
MATLAB code:

```matlab
function df = boundary_layer(eta,f)
    df = zeros(3,1);
    df(1) = f(2);
    df(2) = f(3);
    df(3) = -f(1)*f(3);
end
```

%I would recommend trying these codes in separate .m files, but you can run them in the same .m file if you define the boundary_layer function after the shooting method script. MATLAB will see that ODE45 is taking in a function called boundary_layer and will look for it later in the .m file.

```matlab
%this code uses the shooting method to transform the given boundary value %problem into an initial value problem...strictly speaking, f(0) = 0, %f'(0) = 0, and f'(inf) = 1 are all boundary values. However, ODE45 does %not know the difference between eta and time, it's just an arbitrary name. %Therefore, if we think of eta as a time variable, then we only must %transform one of these conditions. That is, ode45 will think of f(0) and %f'(0) as initial values, but then it will also demand f'(0), which we %don't know. Solution, guess values of f'(0) until f'(inf) == 1. We'll %start with a guess of f'(0) == 0, and keeping updating our guess until %f'(inf) is sufficiently close to 1. Note, this code only integrates from %eta equals 0 to 10. So how is f'(eta = inf) checked? It turns out, f'(eta) %converges very quickly to its final value, so that checking f'(10) is %sufficient for checking f'(inf). We'll check the last value in the f' column, %since that will be f'(10). Once we find a good initial guess for f'(0), %we use that guess and plot the solution.

count = 1;
bestguess = 0;
while (true == 1)
    [eta,I] = ode45(@(boundary_layer,[0,10],[0,0,.0001*count]));
    if abs(f(length(f),2) -1) < .0001
        true = 0;
        sprintf('count = %d',count)
        bestguess = count*.0001;
        break
    else
        count = count + 1;
    end
end
```
\[ \text{[eta,f]} = \text{ode45}(@\text{boundary\_layer},\lbrack 0,10\rbrack,\lbrack 0,0,\text{bestguess}]) \]

\text{hold on}
\text{plot(eta,f(:,1),’b--‘)}
\text{plot(eta,f(:,2),’r‘)}
\text{plot(eta,f(:,3),’g–‘)}
\text{title(’f, f’, and f’’’ vs. \text{eta}')}
\text{xlabel(’\text{eta}')}
\text{legend(’f(\text{eta})’,’f’(\text{eta})’,’f’’’(\text{eta})’)}
4) Drag on a plate

The flow velocity in the boundary layer is given by
\[ u = \frac{\partial \Psi}{\partial y}, \quad v = -\frac{\partial \Psi}{\partial x}, \text{ where } \Psi = (2vUx)^{1/2} f(\eta), \text{ with } \eta = \frac{y}{(2v/U)^{1/2}}. \]

a) Find the shear stress along the plate.
b) Using the results from part a) and also the numerical solution from question 3 to determine the drag on a plate of length L.

Solution:
a) The general form of shear stress in 2 dimensions is given by:
\[ \tau = \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \]

Substituting in the expressions for u and v in terms of the stream function, we have:
\[ \tau = \mu \left( \frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2} \right) \]

We will evaluate the two derivative terms using a mixture of chain rule and produce rule.

\[ \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \sqrt{2vUx} \cdot \frac{\partial f}{\partial \eta} \cdot \frac{1}{\sqrt{2v/\eta}} = U f'(\eta) \]

\[ \frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial}{\partial y} \left( U f'(\eta) \right) = U f''(\eta) \frac{\partial \eta}{\partial y} = U f''(\eta) \cdot \frac{U}{\sqrt{2vx}} \]

\[ \frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x} \left( \sqrt{2vUx} \cdot f(\eta) \right) = \frac{\partial}{\partial x} \left( \sqrt{2vUx} \cdot f(\eta) \right) + \left( \sqrt{2vUx} \right) \cdot \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} \]

\[ = \frac{\sqrt{2vU}}{2x^{1/2}} \cdot f(\eta) + \left( \sqrt{2vUx} \right) \cdot f'(\eta) \cdot \frac{-y}{2 \sqrt{2v/\eta} x^{3/2}} \]
\[
\frac{\partial^2 \psi}{\partial x^2} = C'(x) \cdot f(\eta) + C(x) \cdot f'(\eta) \cdot \frac{\partial \eta}{\partial x} + B'(x, y) \cdot f'(\eta) + B(x, y) \cdot f''(\eta) \cdot \frac{\partial \eta}{\partial x}
\]

So that, \(\tau(\eta, x, y)\) is:

\[
\tau = \mu \left( U f''(\eta) \cdot \sqrt{\frac{U}{2vx}} + \frac{\sqrt{2vU}}{4x^2} \cdot f(\eta) + \frac{\sqrt{2vU}}{2x^2} \cdot f'(\eta) \cdot \frac{y}{2\sqrt{\frac{2vU}{x^2}}} - \frac{Uy}{2x} \cdot f''(\eta) \cdot \frac{y}{2\sqrt{\frac{2vU}{x^2}}} \right)
\]

The shear stress along the plate is given by \(\tau_w(\eta, x, y) = \tau(0, x, 0)\) since, both \(y\) and \(\eta\) are zero at the wall. Substituting in values from Problem 3 we have:

\[
\tau_w(\eta, x, y) = \mu U f''(\eta) \cdot \sqrt{\frac{U}{2vx}} = \mu \frac{4696U^{3/2}}{\sqrt{2vx}}
\]

b) The drag on the plate is the integral of the wall shear stress over the area:

\[
D = \int_0^L \int_0^S \mu \frac{4696U^{3/2}}{\sqrt{2vx}} \, dx \, dz
\]

Note: Since we have assumed flow in the x and y plane, where the x coordinate is along the plate, the z coordinate is perpendicular to the x and y component and points out of the paper. This is a minor detail. If you assumed a Span S of unity, then the double integral becomes a single integral and gives the drag per unit span. In addition, Acheson gives a pre-factor of 2 in front of this integral. This 2 takes into account flow over both sides of the plate. I will drop that 2 for this integration, assuming flow is only over one side of the plate. Solutions in any of the above forms will receive full credit.
\[ D = \frac{4696U^2 \mu \sqrt{2}}{\sqrt{\nu}} \int_0^S dz \cdot 2\sqrt{L} = \frac{4696U^2 \mu \sqrt{2}}{\sqrt{\nu}} \cdot \sqrt{2L} \cdot S \]