HW6 Solutions

2) Boundary Layer equations in non-dimensional forms

Rewrite the exact 2D equations of motion in terms of the non-dimensional and scaled variables

$$x' = \frac{x}{L}, \quad y' = \frac{y}{\mathrm{Re}^{-1/2} L}, \; u' = \frac{u}{U_0}, \; v' = \frac{v}{\mathrm{Re}^{-1/2} U_0}, \; p' = \frac{p}{\rho U_0^2},$$

where $\mathrm{Re} = U_0 L / \nu$. By taking the limit $\mathrm{Re} \to \infty$ with fixed $u', \partial u' / \partial x'$, etc.. derive the boundary layer equations in their non-dimensional and scaled form in steady state:

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2},$$

$$0 = -\frac{\partial p'}{\partial y'}, \; \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0 .$$

Solution:

This problem is an exercise in non-dimensionalization. Chain-rule will be used heavily. Our task is to re-write the steady 2-D Navier-Stokes Equation and Continuity equation in a non-dimensionalized form, take the limit as Reynolds number goes to infinity, and thereby derive the above non-dimensionalized boundary layer equations.

Incompressible Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2-D steady Navier-Stokes Equation:

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rho \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)$$

We have explicit expressions of u and v and in terms of non-dimensional velocities u' and v', but the derivative terms are a little more tricky. As a goal, we want to write each dimensional derivative in terms of a non-dimensional derivative. So for example, let's write $\frac{\partial u}{\partial x}$ in terms of $\frac{\partial u'}{\partial x'}$. Using chain rule we have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial u'} \cdot \frac{\partial u'}{\partial x'} \cdot \frac{\partial x'}{\partial x} = \left(\frac{U_o}{L}\right)\frac{\partial u'}{\partial x'}$$

The 2nd derivatives are a little tricky:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{U_o}{L}\frac{\partial u'}{\partial x'}\right) = \frac{U_o}{L}\frac{\partial u'_{x'}}{\partial x} = \frac{U_o}{L}\frac{\partial^2 u'}{\partial x'^2} \cdot \frac{1}{L} = \left(\frac{U_o}{L^2}\right)\frac{\partial^2 u'}{\partial x'^2}$$

Here, $\frac{\partial u'}{\partial x'}$ was replaced with $u'_{x'}$, read as the derivative of u' with respect to x'. This may make the notation a little simpler to follow. Applying the same process to the other derivative terms in the 2-D steady Navier-Stokes Equation, we find:

$$\frac{\partial u}{\partial y} = \left(\frac{U_o}{Re^{-\frac{1}{2}}\,L}\right)\frac{\partial u'}{\partial y'} \quad and \quad \frac{\partial^2 u}{\partial y^2} = \left(\frac{U_o}{Re^{-1}\,L^2}\right)\frac{\partial^2 u'}{\partial y'^2}$$

$$\frac{\partial v}{\partial x} = \left(\frac{Re^{-\frac{1}{2}}\,U_o}{L}\right)\frac{\partial v'}{\partial x'} \quad and \quad \frac{\partial^2 v}{\partial x^2} = \left(\frac{Re^{-\frac{1}{2}}\,U_o}{L^2}\right)\frac{\partial^2 v'}{\partial x'^2}$$

$$\frac{\partial v}{\partial y} = \left(\frac{U_o}{L}\right)\frac{\partial v'}{\partial y'} \quad and \quad \frac{\partial^2 v}{\partial y^2} = \left(\frac{U_o}{Re^{-\frac{1}{2}}\,L^2}\right)\frac{\partial^2 v'}{\partial y'^2}$$

$$\frac{\partial p}{\partial x} = \left(\frac{\rho {U_o}^2}{L}\right)\frac{\partial p'}{\partial x'} \quad and \quad \frac{\partial p}{\partial y} = \left(\frac{\rho {U_o}^2}{Re^{-\frac{1}{2}}\,L}\right)\frac{\partial p'}{\partial y'}$$

Using these derivative expressions and the definitions of non-dimensional quantities given in the original problem statement, plug these into the continuity equation and 2-D steady Navier-Stokes Equations:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\left[\left(\frac{U_o}{L}\right)\frac{\partial u'}{\partial x'}\right] + \left[\left(\frac{U_o}{L}\right)\frac{\partial v'}{\partial y'}\right] = 0$$

$$\therefore \quad \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

x-momentum:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(U_o u' \cdot \left[\left(\frac{U_o}{L}\right)\frac{\partial u'}{\partial x'}\right] + Re^{-\frac{1}{2}}U_o v' \cdot \left[\left(\frac{U_o}{Re^{-\frac{1}{2}}L}\right)\frac{\partial u'}{\partial y'}\right]\right) = -\left(\frac{\rho U_o{}^2}{L}\right)\frac{\partial p'}{\partial x'} + \mu\left(\left[\left(\frac{U_o}{L^2}\right)\frac{\partial^2 u'}{\partial x'^2}\right] + \left[\left(\frac{U_o}{Re^{-1}L^2}\right)\frac{\partial^2 u'}{\partial y'^2}\right]\right)$$

$$\frac{\rho U_o{}^2}{L}\left(u' \cdot \frac{\partial u'}{\partial x'} + v' \cdot \frac{\partial u'}{\partial y'}\right) = -\left(\frac{\rho U_o{}^2}{L}\right)\frac{\partial p'}{\partial x'} + \frac{\rho U_o{}^2}{L}\left(\left[\left(\frac{\mu}{\rho L U_o}\right)\frac{\partial^2 u'}{\partial x'^2}\right] + \left[\left(\frac{1}{Re^{-1}\cdot Re}\right)\frac{\partial^2 u'}{\partial y'^2}\right]\right)$$

$$u' \cdot \frac{\partial u'}{\partial x'} + v' \cdot \frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \left(\frac{1}{Re}\right)\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}$$

Taking the limit as $\boldsymbol{Re} -> \infty$, we have:

$$\therefore \quad u' \cdot \frac{\partial u'}{\partial x'} + v' \cdot \frac{\partial u'}{\partial y'} = -\frac{\partial p'}{\partial x'} + \frac{\partial^2 u'}{\partial y'^2}$$

y-momentum:

$$\rho\left(u\frac{\partial v}{\partial x}+\ v\frac{\partial v}{\partial y}\right)=\ -\frac{\partial p}{\partial y}+\ \mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\rho\left(U_o u'\cdot\left[\left(\frac{Re^{-\frac{1}{2}}U_o}{L}\right)\frac{\partial v'}{\partial x'}\right]+\ Re^{-\frac{1}{2}}U_o v'\cdot\left(\frac{U_o}{L}\right)\frac{\partial v'}{\partial y'}\right)=\ -\left(\frac{\rho {U_o}^2}{Re^{-\frac{1}{2}}L}\right)\frac{\partial p'}{\partial y'}+\ \mu\left(\left[\left(\frac{Re^{-\frac{1}{2}}U_o}{L^2}\right)\frac{\partial^2 v'}{\partial x'^2}\right]+\left[\left(\frac{U_o}{Re^{-\frac{1}{2}}L^2}\right)\frac{\partial^2 v'}{\partial y'^2}\right]\right)$$

$$\frac{\rho {U_o}^2}{L}Re^{-\frac{1}{2}}\left(u'\cdot\frac{\partial v'}{\partial x'}+\ v'\cdot\frac{\partial v'}{\partial y'}\right)=\ -\left(\frac{\rho {U_o}^2}{Re^{-\frac{1}{2}}L}\right)\frac{\partial p'}{\partial y'}+\ \frac{\rho {U_o}^2}{L}\left(\left[\left(\frac{Re^{-\frac{1}{2}}\mu}{\rho U_o L}\right)\frac{\partial^2 v'}{\partial x'^2}\right]+\left[\left(\frac{1}{Re^{1/2}}\right)\frac{\partial^2 v'}{\partial y'^2}\right]\right)$$

$$Re^{-\frac{1}{2}}\left(u'\cdot\frac{\partial v'}{\partial x'}+\ v'\cdot\frac{\partial v'}{\partial y'}\right)=\ -\left(\frac{1}{Re^{-1/2}}\right)\frac{\partial p'}{\partial y'}+\ \left(\left(\frac{1}{Re^{3/2}}\right)\frac{\partial^2 v'}{\partial x'^2}+\left[\left(\frac{1}{Re^{1/2}}\right)\frac{\partial^2 v'}{\partial y'^2}\right]\right)$$

Taking the limit as $\boldsymbol{Re} -> \infty$, we have:

$$\therefore\ 0=\ -\frac{\partial p'}{\partial y'}$$

3) Matlab solution of the similarity equation

For flow past a plate, the similarity solution $f(\eta)$ is governed by
$f''' + ff'' = 0$, with boundary conditions $f(0) = f'(0) = 0, f'(\infty) = 1$.
Solve $f(\eta)$ numerical using Matlab and plot your solution.
[Hint: you can define $g(\eta) = f'(\eta)$, $h(\eta) = g'(\eta)$ to reduce the 3rd order differential equation into three first order differential equations.]

Solution:

MATLAB code:

```
%set up system of 3 1st order ODEs, system will be passed into ODE45
function df = boundary_layer(eta,f)
df = zeros(3,1);
df(1) = f(2);
df(2) = f(3);
df(3) = -f(1)*f(3);
end
```

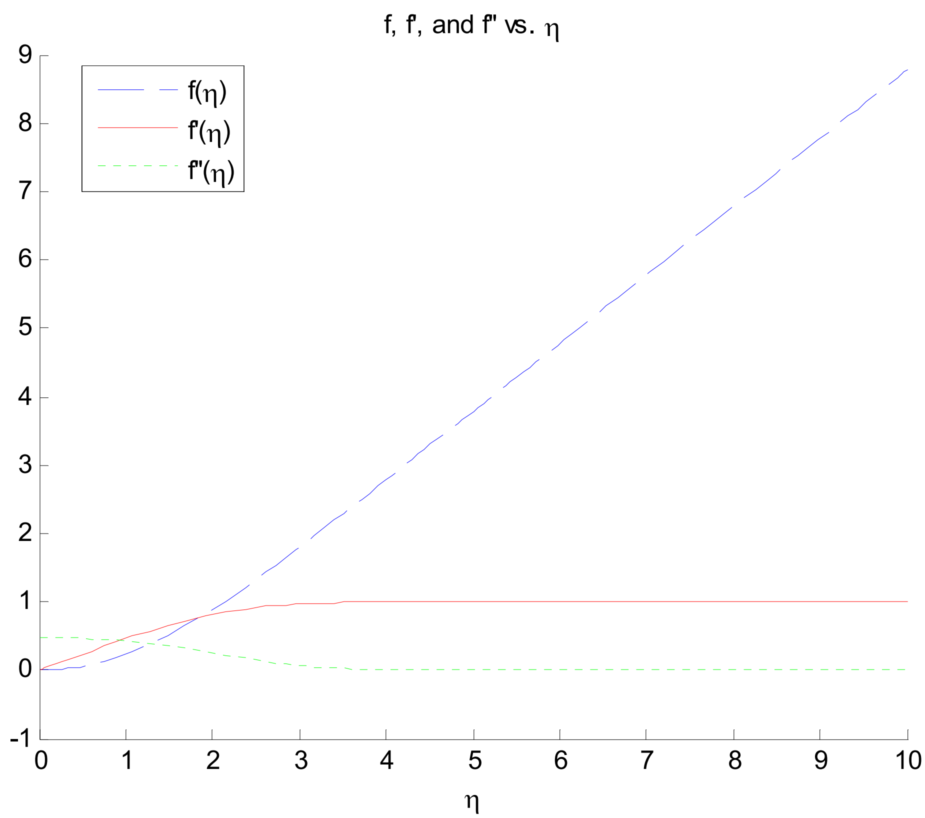
%I would recommend trying these codes in separate .m files, but you can run them in the same .m file if you define the boundary_layer function after the shooting method script. MATLAB will see that ODE45 is taking in a function called boundary_layer and will look for it later in the .m file.

```
%this code uses the shooting method to transform the given boundary value
%problem into an initial value problem...strictly speaking, f(0)= 0,
%f'(0) = 0, and f'(inf) = 1 are all boundary values. However, ODE45 does
%not know the difference between eta and time, it's just an arbitrary name.
%Therefore, if we think of eta as a time variable, then we only must
%transform one of these conditions. That is, ode45 will think of f(0) and
%f'(0) as initial values, but then it will also demand f''(0), which we
%don't know. Solution, guess values of f''(0) until f'(inf) == 1. We'll
%start with a guess of f''(0) == 0, and keeping updating our guess until
%f'(inf) is sufficiently close to 1. Note, this code only integrates from
%eta equals 0 to 10. So how is f'(eta = inf) checked? It turns out, f'(eta)
%converges very quickly to its final value, so that checking f'(10) is
%sufficient for checking f'(inf). We'll check the last value in the f' column,
%since that will be f'(10). Once we find a good initial guess for f''(0),
%we use that guess and plot the solution.

count = 1;
bestguess = 0;
while (true == 1)
    [eta,f] = ode45(@boundary_layer,[0,10],[0,0,.0001*count]);
    if abs(f(length(f),2) -1) < .0001
        true = 0;
        sprintf('count = %d',count)
        bestguess = count*.0001;
        break
    else
        count = count + 1;
    end
end
```

```
[eta,f] = ode45(@boundary_layer,[0,10],[0,0,bestguess]);

hold on
plot(eta,f(:,1),'b--')
plot(eta,f(:,2),'r')
plot(eta,f(:,3),'g:')
title('f, f'', and f'''' vs. \eta')
xlabel('\eta')
legend('f(\eta)','f''(\eta)','f''''(\eta)')
```



4) Drag on a plate

The flow velocity in the boundary layer is given by
$u = \frac{\partial \Psi}{\partial y}$, $v = -\frac{\partial \Psi}{\partial x}$, where $\Psi = (2\nu U x)^{1/2} f(\eta)$, with $\eta = \frac{y}{(2\nu x / U)^{1/2}}$.

a) Find the shear stress along the plate.
b) Using the results from part a) and also the numerical solution from question 3 to determine the drag on a plate of length L.

Solution:
a) The general form of shear stress in 2 dimensions is given by:

$$\tau = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Substituting in the expressions for u and v in terms of the stream function, we have:

$$\tau = \mu\left(\frac{\partial^2 \Psi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x^2}\right)$$

We will evaluate the two derivative terms using a mixture of chain rule and produce rule.

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \sqrt{2\upsilon U x} \cdot \frac{\partial f}{\partial \eta} \cdot \frac{1}{\sqrt{\frac{2\upsilon x}{U}}} = U f'(\eta)$$

$$\frac{\partial^2 \Psi}{\partial y^2} = \frac{\partial}{\partial y}\left(U f'(\eta)\right) = U f''(\eta)\frac{\partial \eta}{\partial y} = \boldsymbol{U f''(\eta) \cdot \sqrt{\frac{U}{2\upsilon x}}}$$

$$\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{2\upsilon U x} \cdot f(\eta)\right) = \frac{\partial}{\partial x}\left(\sqrt{2\upsilon U x}\right) \cdot f(\eta) + \left(\sqrt{2\upsilon U x}\right) \cdot \frac{\partial f}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$= \frac{\sqrt{2\upsilon U}}{2x^{1/2}} \cdot f(\eta) + \left(\sqrt{2\upsilon U x}\right) \cdot f'(\eta) \cdot \frac{-y}{2\sqrt{\frac{2\upsilon}{U}}\, x^{3/2}}$$

$$= C(x)\cdot f(\eta) + B(x,y)\cdot f'(\eta)$$

$$\frac{\partial^2\Psi}{\partial x^2} = C'(x)\cdot f(\eta) + C(x)\cdot f'(\eta)\cdot\frac{\partial\eta}{\partial x} + B'(x,y)\cdot f'(\eta) + B(x,y)\cdot f''(\eta)\cdot\frac{\partial\eta}{\partial x}$$

$$\frac{\partial^2\Psi}{\partial x^2} = -\frac{\sqrt{2vU}}{4x^{3/2}}\cdot f(\eta) + \frac{\sqrt{2vU}}{2x^{1/2}}\cdot f'(\eta)\cdot\frac{-y}{2\sqrt{\frac{2v}{U}}x^{3/2}} + \frac{Uy}{2x^2}\cdot f'(\eta) + -\frac{Uy}{2x}\cdot f''(\eta)\cdot\frac{-y}{2\sqrt{\frac{2v}{U}}x^{3/2}}$$

So that, $\tau(\eta,x,y)$ is:

$$\tau = \mu\left(Uf''(\eta)\cdot\sqrt{\frac{U}{2vx}} + \frac{\sqrt{2vU}}{4x^{\frac{3}{2}}}\cdot f(\eta) + \frac{\sqrt{2vU}}{2x^{\frac{1}{2}}}\cdot f'(\eta)\cdot\frac{y}{2\sqrt{\frac{2v}{U}}x^{\frac{3}{2}}} - \frac{Uy}{2x^2}\cdot f'(\eta) - \frac{Uy}{2x}\cdot f''(\eta)\cdot\frac{y}{2\sqrt{\frac{2v}{U}}x^{\frac{3}{2}}}\right)$$

The shear stress along the plate is given by $\tau_w(\eta,x,y) = \tau(0,x,0)$ since, both y and η are zero at the wall. Substituting in values from Problem 3 we have:

$$\tau_w(\eta,x,y) = \mu U f''(\eta)\cdot\sqrt{\frac{U}{2vx}} = \mu\frac{.4696U^{3/2}}{\sqrt{2vx}}$$

b) The drag on the plate is the integral of the wall shear stress over the area:

$$D = \int_0^S\int_0^L \mu\frac{.4696U^{3/2}}{\sqrt{2vx}}dxdz$$

Note: Since we have assumed flow in the x and y plane, where the x coordinate is along the plate, the z coordinate is perpendicular to the x and y component and points out of the paper. This is a minor detail. If you assumed a Span S of unity, then the double integral becomes a single integral and gives the drag per unit span. In addition, Acheson gives a pre-factor of 2 in front of this integral. This 2 takes into account flow over both sides of the plate. I will drop that 2 for this integration, assuming flow is only over one side of the plate. Solutions in any of the above forms will receive full credit.

$$D = \int_0^S\int_0^L \mu\frac{.4696U^{\frac{3}{2}}}{\sqrt{2vx}}dxdz = \frac{.4696U^{\frac{3}{2}}\mu}{\sqrt{2v}}\int_0^S dz\int_0^L\frac{dx}{\sqrt{x}}$$

$$\therefore\ D = \frac{.4696U^{\frac{3}{2}}\mu}{\sqrt{2v}}\int_0^S dz\ \cdot 2\sqrt{L} = \frac{\mathbf{.4696}\boldsymbol{U}^{\frac{\mathbf{3}}{\mathbf{2}}}\boldsymbol{\mu}}{\sqrt{\boldsymbol{v}}}\cdot\sqrt{\mathbf{2}\boldsymbol{L}}\cdot\boldsymbol{S}$$