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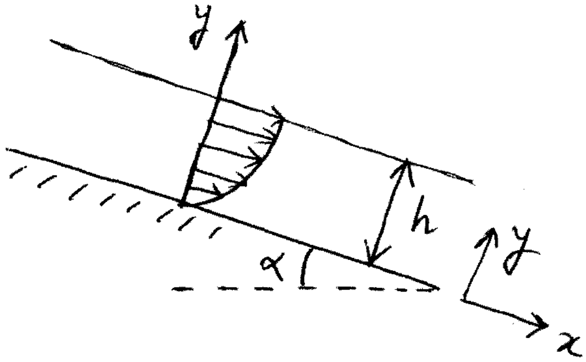
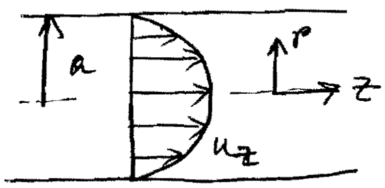
3/1/11

MAE 4230/5230
SOLUTION TO HOMEWORK 4

⟨2⟩ Relation between flow under gravity down an inclined plane and 1-D pipe flow





$$u = \frac{g}{2\nu} y(2h - y)\sin\alpha$$
$$v = w = 0$$

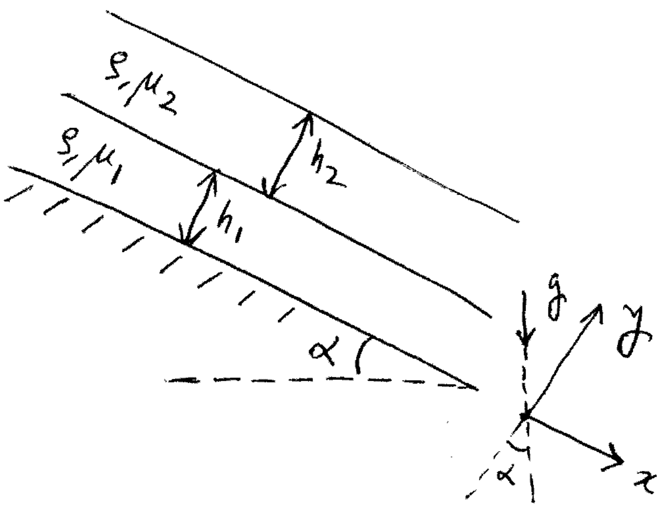
$$u_z = \frac{P}{4\mu}(a^2 - r^2)$$
$$u_r = u_\theta = 0$$

1. $u = u_{max}$ at $y = h$
2. $\tau = \mu \frac{du}{dy} = 0$ at $y = h$

1. $u_z = u_{z,max}$ at $r = 0$
2. $\tau_{zr} = \mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) = \mu \frac{du_z}{dr} = 0$ at $r = 0$

Therefore, the flow down an inclined plane, viewed in the x-y plane, resembles half of a 1-D pipe flow (viewed in the r-z plane). In addition, both flows are steady, 1-D, incompressible and fully-developed. However, 1-D pipe flow is driven by an external pressure gradient while gravity drives the flow down an inclined plane.

⟨3⟩ (2.4 in Acheson)



Steady, fully developed, 1-D, incompressible flow

Here, we have two fluids and we have to go back and forth between the lower and upper fluids to determine our unknown constants.

<u>Lower fluid</u> : $\underline{u} = [u_1(y), 0, 0]$

Navier-Stokes equations under the conditions of the problem reduce to :

x-component : $0 = -\frac{1}{\rho}\frac{\partial P_1}{\partial x} + \nu_1 \frac{\partial^2 u_1}{\partial y^2} + g\sin\alpha$

y-component : $0 = -\frac{1}{\rho}\frac{\partial P_1}{\partial y} + 0 - g\cos\alpha$

z-component : $0 = -\frac{1}{\rho}\frac{\partial P_1}{\partial z} + 0 + 0$

$\therefore\ P_1 = P_1(x, y)$

$\therefore\ \frac{\partial P_1}{\partial y} = -\rho g\cos\alpha \Rightarrow \underline{P_1 = -(\rho g\cos\alpha)\, y + C_1(x)}$ — (1)

Upper fluid : $\underline{u} = [u_2(y), 0, 0]$

Navier-Stokes equations under the conditions of the problem reduce to :

x-component : $0 = -\frac{1}{\rho}\frac{\partial P_2}{\partial x} + \nu_2 \frac{\partial^2 u_2}{\partial y^2} + g\sin\alpha$

y-component : $0 = -\frac{1}{\rho}\frac{\partial P_2}{\partial y} + 0 - g\cos\alpha$

z-component : $0 = -\frac{1}{\rho}\frac{\partial P_2}{\partial z} + 0 + 0$

$\therefore P_2 = P_2(x, y)$

$\therefore \frac{\partial P_2}{\partial y} = -\rho g\cos\alpha \Rightarrow P_2 = -\rho g\cos\alpha)y + C_2(x)$

B.C. : at $y = h_1 + h_2$, $P_2 = P_0$ (atmospheric pressure)

$\Rightarrow P_0 = -\rho g\cos\alpha(h_1+h_2) + C_2(x)$

$\therefore P_2 = P_2(y) = P_0 + \rho g\cos\alpha(h_1+h_2-y)$ — (2)

$\therefore \frac{\partial P_2}{\partial x} = 0 \Rightarrow$ flow is driven by gravity only down the inclined plane

$\therefore \frac{\partial^2 u_2}{\partial y^2} = -\frac{g\sin\alpha}{\nu_2}$

$\Rightarrow \frac{\partial u_2}{\partial y} = -\frac{g\sin\alpha}{\nu_2}y + C_5$ — (6)

B.C. : at $y = h_1 + h_2$, $\mu_2 \dfrac{du_2}{dy} = 0$

(zero stress at the free surface)

$$\Rightarrow \quad -\frac{g\sin\alpha}{\nu_2}(h_1+h_2) + C_5 = 0$$

$$\Rightarrow \quad C_5 = \frac{g\sin\alpha}{\nu_2}(h_1+h_2)$$

$$\therefore \quad \frac{du_2}{dy} = \frac{g\sin\alpha}{\nu_2}(h_1+h_2-y) \qquad \text{— (7)}$$

Lower fluid (Contd.) :

From upper fluid analysis [equation 2], we know $P_2(y)$

B.C. : at $y = h_1$, $P_1(h_1) = P_2(h_1)$ [Pressure continuity at the interface]

$$\Rightarrow \quad -\rho g\cos\alpha\,(h_1) + c_1(x) = P_0 + \rho g\cos\alpha\,(h_2)$$

$$\Rightarrow \quad c_1(x) = P_0 + \rho g\cos\alpha\,(h_1 + h_2)$$

$$\therefore \quad P_1 = P_1(y) = P_0 + \rho g\cos\alpha\,(h_1 + h_2 - y) \quad \text{— (3)}$$

From (3), $\dfrac{\partial P_1}{\partial x} = 0$ i.e., the flow is driven by gravity down the incline.

$$\therefore \quad \frac{\partial^2 u_1}{\partial y^2} = -\frac{g\sin\alpha}{\nu_1}$$

$$\Rightarrow \quad \frac{\partial u_1}{\partial y} = -\frac{g\sin\alpha}{\nu_1}\,y + C_3 \quad \text{— (4)}$$

$$\therefore \quad u_1 = -\frac{g\sin\alpha}{\nu_1}\left(\frac{y^2}{2}\right) + C_3 y + C_4$$

B.C. : at $y = 0$, $u_1 = 0$

$$\Rightarrow \quad 0 = C_4$$

$$\therefore \quad u_1 = -\frac{g\sin\alpha}{\nu_1}\left(\frac{y^2}{2}\right) + C_3 y \quad \text{— (5)}$$

B.C.: at $y = h_1$, $\mu_1 \frac{du_1}{dy} = \mu_2 \frac{du_2}{dy}$ [Newton's third law; shear stress balance at the interface]

Using equation (7) from upper fluid analysis,

$$-\mu_1 \frac{g\sin\alpha}{\nu_1} y + \mu_1 C_3 = \mu_2 \frac{g\sin\alpha}{\nu_2} (\cancel{h_1} + h_2 - \cancel{y})$$

$$\Rightarrow -\rho g\sin\alpha (h_1) + \mu_1 C_3 = \rho g \sin\alpha (h_2)$$

$$\Rightarrow \quad C_3 = \frac{g\sin\alpha}{\nu_1}(h_1 + h_2)$$

Putting in (5), we get the velocity of the lower fluid as

$$u_1 = -\frac{g\sin\alpha}{\nu_1}\left(\frac{y^2}{2}\right) + \frac{g\sin\alpha}{\nu_1}(h_1+h_2)y$$

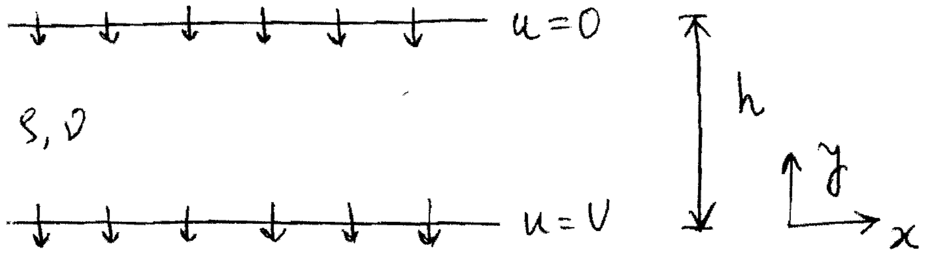
$$\Rightarrow \boxed{u_1(y) = \left[(h_1+h_2)y - \frac{1}{2}y^2\right]\frac{g\sin\alpha}{\nu_1}}$$

* In the above analysis, we find that μ_2 falls out of the expression for u_1 when we apply the shear stress balance at the interface. Consider the situation when we only have the lower fluid (μ_1). Then, at the free surface $y = h_1$, $\mu_1 \frac{\partial u_1}{\partial y} = 0$. Now, let us ~~put~~ introduce

the upper fluid (μ_2) on top. Now, at $y=h_1$, $\mu_1 \frac{\partial u_1}{\partial y}$ is no longer zero but modified to be equal to $\mu_2 \frac{\partial u_2}{\partial y}$, where u_2 now is such that at $y=h_1+h_2$, $\mu_2 \frac{\partial u_2}{\partial y} = 0$. Thus, it turns out that u_1 doesnot depend on μ_2 due to the balance of stresses at the interface but depends on h_2 due to the stress-free boundary condition at the free surface (which can be thought of as being shifted from $y=h_1$ to $y=h_1+h_2$ by addition of the fluid on top).

* Another way to look at it: The shear stress at the interface on the lower fluid is the result of a component of the weight of the upper fluid, which depends on the volume of the upper fluid (and hence on h_2) but not on μ_2. That is why, we find that $\mu_2 \frac{\partial u_2}{\partial y}$ at $y=h_1$ is independent of μ_2 but dependent on h_2. Note that u_1 could potentially depend on ρ_2 ($=\rho_1$ here), which supports this argument.

⟨4⟩ (2.6 in Acheson)



Steady, fully developed, 2-D, incompressible flow

$\underline{u} = [u(y), v(y), 0]$

<u>continuity</u> : $\underline{\nabla} . \underline{u} = 0 \Rightarrow \cancel{\frac{\partial u}{\partial x}}^{0} + \frac{\partial v}{\partial y} = 0$

$\Rightarrow \frac{\partial v}{\partial y} = 0$

$\Rightarrow v \neq v(y) = \text{constant}$

But, at $y = 0$, $v = -v_0$

$\therefore \boxed{v = -v_0}$ for all y i.e. throughout the channel

$\therefore \underline{u} = [u(y), -v_0, 0]$

<u>Momentum (Navier-Stokes)</u> :

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u} . \underline{\nabla}) \underline{u} = -\frac{1}{\rho} \underline{\nabla} p + \nu \nabla^2 \underline{u}$$

<u>x-component</u> : $\cancel{\frac{\partial u}{\partial t}}^{0} + u \cancel{\frac{\partial u}{\partial x}}^{0} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{0} + \frac{\partial^2 u}{\partial y^2}\right)$

$$\Rightarrow \quad -v_0 \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \quad \text{— (1)}$$

Here, the flow is driven by the bottom plate moving with a velocity $u = U$. Therefore, we can assume no pressure gradient i.e. $\frac{\partial P}{\partial x} = 0$

$\therefore$ Equation (1) reduces to

$$\boxed{\frac{d^2u}{dy^2} + \left(\frac{v_0}{\nu}\right)\frac{du}{dy} = 0} \quad \text{—} (2)$$

Equation (2) is a 2nd order linear ODE with constant coefficients. Hence, we try the solution $u(y) = e^{ry}$, where r is a constant.

Putting in (2),

$$r^2 e^{ry} + \left(\frac{v_0}{\nu}\right) r e^{ry} = 0$$

$$\Rightarrow r^2 + \left(\frac{v_0}{\nu}\right) r = 0$$

$$\Rightarrow r\left[r + v_0/\nu\right] = 0$$

$$\therefore r_1 = 0;\ r_2 = -(v_0/\nu)$$

$\therefore$ The general solution is given by :

$$u(y) = c_1 e^{r_1 y} + c_2 e^{r_2 y}$$
$$= c_1 + c_2 \exp\left(-\frac{v_0}{\nu} y\right)$$

Boundary conditions :

(i) at $y = 0$, $u = U \Rightarrow$ $U = c_1 + c_2$ — (3)

(ii) at $y = h$, $u = 0$ $\Rightarrow$ $0 = c_1 + c_2 \exp(-\frac{v_0}{\nu} h)$

$\therefore$ ③ $-$ ④ gives, $U = c_2 [1 - \exp(-\frac{v_0}{\nu} h)]$

$$\Rightarrow \quad c_2 = \frac{U}{1 - \exp(-\frac{v_0}{\nu} h)}$$

$$\therefore \quad c_1 = -c_2 \exp(-\frac{v_0}{\nu} h) = -\frac{U \exp(-\frac{v_0}{\nu} h)}{1 - \exp(-\frac{v_0}{\nu} h)}$$

$$\therefore \quad u(y) = c_1 + c_2 \exp(-\frac{v_0}{\nu} y)$$

$$= \frac{-U \exp(-\frac{v_0}{\nu} h) + U \exp(-\frac{v_0}{\nu} y)}{1 - \exp(-\frac{v_0}{\nu} h)}$$

$$= U \left[\frac{\exp(-\frac{v_0}{\nu} y) - \exp(-\frac{v_0}{\nu} h)}{1 - \exp(-\frac{v_0}{\nu} h)} \right]$$

$\therefore$ The velocity field is given by

$$\boxed{\underline{u} = \left[U \left\{ \frac{\exp(-\frac{v_0}{\nu} y) - \exp(-\frac{v_0}{\nu} h)}{1 - \exp(-\frac{v_0}{\nu} h)} \right\}, -v_0, 0 \right]}$$

If the velocity profile within the channel is to be similar to a boundary layer, then for some $y < h$, the velocity u should reduce to 1% of the boundary velocity U.

Let $\frac{v_0 h}{\nu} = M \gg 1$ and let us now find y for which $u = 1\%$ of $U = 0.01\,U$

$$\therefore \quad 0.01\,U = U\left[\frac{\exp\left[-\frac{v_0}{\nu}y\right] - \cancel{\exp[-M]}^{\,0}}{1 - \cancel{\exp[-M]}^{\,0}}\right]$$

$$\Rightarrow \quad 0.01 = \exp\left(-\frac{v_0}{\nu}y\right)$$

$$\Rightarrow \quad \ln(0.01) = -\frac{v_0}{\nu}\cdot y$$

$$\Rightarrow \quad y = \frac{4.605\,\nu}{v_0} = \left(\frac{4.605}{M}\right)h$$

$\therefore$ If $M \gg 1$, $y \ll h$. Also, $y \propto \nu$. Therefore, we conclude that if $\frac{v_0 h}{\nu} \gg 1$, the velocity profile in the channel resembles a boundary layer with thickness proportional to the kinematic viscosity ν. Also, larger the value of $\frac{v_0 h}{\nu}$, thinner is the boundary layer.