MAE 4230/5230 Homework 3[1] Prof. Wang

Due Date: February 16, 2011

1) Linearization and Iteration

Use an iterative technique to solve the nonlinear equation $u^3 - u^4 = -11$. Show your derivation of the linearized equation as well as the results for your iterations with four initial guesses. There are two solutions. Indicate how your initial guess affects the final solution.

2) Finite Difference Methods and Discretization

Assume that you know the values of u(x) at four gridpoints (u_i, u_{i+1}, u_{i+2}, and u_{i+3}), where u_i is at $x = x_i$, u_{i+1} is at $x = x_i+\Delta x$, u_{i+2} is at $x = x_i+2\Delta x$, and u_{i+3} is at $x = x_i+3\Delta x$.

(a) Write u_{i+1}, u_{i+2}, and u_{i+3} as Taylor expansions of u around x_i.

(b) Combine and rearrange these equations to give an expression for $\left.\frac{d^3u}{dx^3}\right|_i$ in terms of u_i, u_{i+1}, u_{i+2}, and u_{i+3}.

(c) Show that this approximation is first-order, i.e., show that the truncation error is $O(\Delta x)$.

3) Application of Finite Difference Methods to Linear and Nonlinear problems

For one-dimensional flow of fluid between two plates oriented normal to the y axis, the Navier–Stokes equations simplify to

$$0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

For simplicity, consider $\frac{dp}{dx} = -1$ (corresponding to flow from left to right) and $\mu = 1$. Assume that the plates are located at y = 1 and y = -1, at which we apply the boundary condition that u = 0.

(a) Solve this equation analytically by treating $\frac{dp}{dx}$ as a constant and integrating the $\frac{\partial^2 u}{\partial y^2}$ term.

(b) Solve this equation numerically on 5-point and 9-point grids by applying the finite-difference method to this equation to get a linearized difference equation at grid point *i* away from the boundary. Note that a second-order difference approximation for the second-derivative is

$$\left(\frac{d^2u}{dx^2}\right)_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} + O(\Delta x^2) \quad (2)$$

i. Assemble the discrete system of equations for the grid into a matrix system of the form

$$Au = b \quad (3)$$

where, for example, for the 5-point grid:

[1] Contributions by B. Kirby, R. Bhaskaran, and S. Santana.

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} \quad (4)$$

and solve this system using MATLAB. You may solve this system using direct inversion of the matrix, or using an iterative technique. If you use an iterative technique, use u = 0 for your initial guess.

ii. Plot the finite-difference solution obtained on the 5- and 9-point grids and compare it with the exact solution. Plot u on the abscissa and y on the ordinate.

(c) Now solve the equation

$$0 = -\frac{dp}{dx} + \mu\frac{\partial^2 u}{\partial y^2} - .4u^2 \quad (5)$$

On the same domain, with same $\frac{dp}{dx}$ and same μ and same boundary conditions and same grids. In this case, the nonlinear term will require that you use an iterative technique. Use $u = 0$ for your initial guess. Physically, the $-0.4u^2$ term corresponds to a retardation body force per unit volume proportional to the kinetic energy per unit volume of the fluid. This is not common with normal fluids, but can be observed with exotic fluids like ferrofluids that respond to magnetic fields. This solution does not allow for trivially simple integration like the first equation, but a numerical solution is still possible. Solve the equation, plot the results and residual, and compare to the solution without the retardation force. What is the effect of the retardation on the velocity profile?

Why does the solution without the retardation force give zero residual after one iteration while the solution with the retardation force has a finite residual? (Stated another way, why is only one matrix inversion required to solve the first problem?) Also, what does this exercise show you about the relative merits of analytical solution of the differential equations vs. numerical solution of the differential equations?