A derivation of the critical damping condition, $k\tau = 1/e$, for equation $\dot{\theta} = -k\theta(t - \tau)$

Z. Jane Wang

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The solution to the time-delayed differential equation $\dot{\theta} = -k\theta(t - \tau)$ depends on two parameters, the gain $k$ and the time-delay $\tau$. The general solution is of the form $e^{\lambda t}$, $\lambda$ being a complex number. Because of the time delay in the right hand side of the equation, $\lambda$ satisfies the transcendal equation, $\lambda = -ke^{-\lambda\tau}$, instead of a simple algebraic equation. There are no simple ways to solve for $\lambda$ explicitly. The numerical solutions for different $k$ were shown in Fig. 11 in our main manuscript. At $k\tau = 1/e$, the solution is critically damped, i.e., it decays to zero in the shortest possible time. This result has been quoted in the literature[1, 2]. However we have not managed to track down the derivation of this condition. One of the referees of our paper pointed out that the derivation is also not in standard textbooks on control theory. In this note I give a derivation of this condition.

To solve for this time-delayed differential equation,

$$\dot{\theta} = -k\theta(t - \tau),$$

(1)

we start by writing $\theta = e^{\lambda t}$, where $\lambda$ is a complex number, $\lambda = a + ib$. Plugging in this to Eq.(1) leads to the eigenvalue equation for $\lambda$:

$$\lambda e^{\lambda t} = -ke^{\lambda(t-\tau)}$$

(2)

$$\lambda = -ke^{-\lambda\tau}$$

(3)

Since Eq. (3) does not admit explicit solutions, the rest of this note is to show how to find the critical condition without using the explicit value or evoking the special Lambert-function.

By definition, the damping solutions correspond to exponentially decaying functions, where $\lambda = a$ and $b = 0$. According to Eq.(3), $a = -ke^{-a\tau}$. Multiply $\tau$ on both sides,
\[ a\tau = -k\tau e^{-a\tau}, \text{ and let } x = a\tau, \text{ we have} \]

\[ xe^x = -k\tau \quad (4) \]

To find solutions to Eq.(4), let’s consider the following graph (Fig. 1). The curve corresponds to \( y = xe^x \). The solutions of \( a \) are at the intersection of this curve and \(-k\tau\).

\[ \begin{align*}
\text{FIG. 1: The solutions to } xe^x &= -k\tau \\
\end{align*} \]

The graph shows the range of \( k\tau \) for which the solution exists. In general, there are a pair of intersection points, \(-a_1\) and \(-a_2\), which become degenerate at the minimum of the curve. The general solution is therefore the linear superposition of the two, \( Ae^{-a_1t} + Be^{-a_2t} \).

For \( a_1 < a_2 \), the solution is dominated by \( Ae^{-a_1t} \).

Our interest is to find the condition for critical damping. By definition, if the solution is of the form of \( e^{-at} \), the fastest decay occurs at the maximum of all possible \( a \). This is equivalent to finding the maximum of all \( a_1 \). Fig. 1 shows that it occurs at the bottom of the curve of \( y = xe^x \).

This turns the problem of finding the minimum of \( a_1 \) into an easier problem of determining the minimum of \( y(x) = xe^x \). The minimum occurs when \( dy/dx = 0, e^x + xe^x = 0 \), which gives \( x_c = -1 \).

Going back to the definition of \( x = a\tau \), we have the critical damping condition, \( a_c = -1/\tau \).

Recall that \( a_c \) also satisfies the eigenvalue equation, \( a_c\tau = -k\tau e^{-a_c\tau} \), we can express the critical condition in the sought form, in terms of \( k \) and \( \tau \),

\[ k\tau = 1/e \quad (5) \]

This completes the derivation. As a final remark, the proportional control law found in our main manuscript is \( \dot{\theta}_B = k\theta_e(t - \tau) = k[\theta_p(t - \tau) - \theta_B(t - \tau)] \), with \( \theta_B \) being the
orientation of the beetle, and $\theta_p$ the angular position of the prey relative to the beetle’s head. In the case where the prey is at rest as considered in Fig. 11 in the main manuscript, the equation becomes $\dot{\theta}_B = -k\theta_B(t - \tau)$, which is the equation considered here.